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DERIVATION OF THE GENERATING FUNCTION FOR THE BERNOULLI POLYNOMIALS

A generating function is a power series whose coefficients are a sequence of interest. A generating function often enables properties of the sequence to be easily exposed. Here we show that

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$ at least when $|t| < 2\pi$ (Note 1). We make use of the facts (shown in an earlier post) that

$$B_0=1 \quad \text{and} \quad \sum_{i=0}^{k}\binom{k+1}{i}B_i=0 \quad \text{and} \quad B_k(x)=\sum_{i=0}^{k}\binom{k}{i}B_i\, x^{k-i}$$

First consider the finite geometric series

$$\sum_{j=0}^{n-1} e^{jt}=\frac{e^{nt}-1}{e^t-1}$$

We also have (if we adopt the convention that $0^0 = 1$)

$$\sum_{j=0}^{n-1} e^{jt}=\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\frac{j^k t^k}{k!}=\sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{j=0}^{n-1} j^k \quad (n \geq 1)$$

(here and in what follows changes in the order of summation are justified by the fact that the series are absolutely convergent - Note 2). Thus we obtain

$$\frac{e^{nt}-1}{e^t-1}=\sum_{j=0}^{n-1} e^{jt}=\sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{j=0}^{n-1} j^k=\sum_{k=0}^{\infty}\left[B_{k+1}(n)-B_{k+1}\right]\frac{t^k}{k+1!}$$

$$\frac{te^{nt}}{e^t-1}-\frac{t}{e^t-1}=\sum_{k=1}^{\infty} B_k(n)\frac{t^k}{k!}-\sum_{k=1}^{\infty} B_k(0)\frac{t^k}{k!}$$

This suggests that

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}=f(x,t)$$

and certainly $f(x,0) = 1 = B_0(x)$ and $\partial f(x,t)/\partial t = x - 1/2 = B_1(x)$ at $t = 0$. Let

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty} C_k(0)\frac{t^k}{k!}$$

$$t=\sum_{j=1}^{\infty}\frac{t^j}{j!}\sum_{k=0}^{\infty} C_k(0)\frac{t^k}{k!}$$

Because both series are absolutely convergent, we can apply Cauchy's theorem for the multiplication of series (Note 3) and obtain (putting $j + k = m$)

$$t+\sum_{s=0}^{\infty}\frac{t^s}{s!}C_s(0)=\sum_{j=0}^{\infty}\sum_{k=0}^{n}\frac{t^{k+j}}{k!\,j!}C_k(0)=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{t^m}{k!\,m-k!}C_k(0)$$

Equating coefficients of t^n $(n > 1)$ we have

$$\frac{C_n(0)}{n!}=\sum_{k=0}^{n}\frac{C_k(0)}{k!\,n-k!} \quad \text{hence} \quad 0=\sum_{k=0}^{n}\binom{n+1}{k}C_k(0) \quad \text{and} \quad C_n(0)=B_n$$

since we have already shown that $C_0(0) = B_0$.

We now have

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} C_j(x) = \frac{te^{xt}}{(e^t - 1)} = e^{xt} \sum_{j=0}^{\infty} \frac{t^j}{j!} B_j = \left[\sum_{i=0}^{\infty} \frac{x^i t^i}{i!} \right] \left[\sum_{j=0}^{\infty} \frac{t^j}{j!} B_j \right]$$

Once more we can use Cauchy's theorem to multiply the series and obtain

$$\sum_{j=0}^{\infty} \frac{t^j}{j!} C_j(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{x^{n-m} t^{n-m}}{n-m!} \frac{x^m}{m!} B_m = \sum_{n=0}^{\infty} \frac{t^{n-m}}{n-m!\, m!} B_m$$

Equating coefficients of like powers of t we have

$$C_n(x) = \sum_{m=0}^{n} \binom{n}{m} x^{n-m} B_m \quad \text{hence} \quad C_n(x) = B_n(x)$$

The generating function enables us to show simply that $B_{2k+1} = 0$ $(k > 0)$. This is necessarily the case if

$$\frac{t}{e^t - 1} - B_1 t = 1 + \sum_{n=2}^{\infty} B_n \frac{t^n}{n!}$$

is an even function of t. This is so since

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t(e^t + 1)}{2(e^t - 1)} = \frac{-t(e^{-t} + 1)}{2(e^{-t} - 1)} = \frac{-t}{e^{-t} - 1} - \frac{t}{2}$$

NOTES

References are to relevant sections of Whittaker and Watson (A Course of Modern Analysis, 4th Edition reprinted, Cambridge University Press).

(1) the radius of convergence of the Taylor series (considered as a function of a complex variable) is at least so large as to exclude from the interior of the circle the nearest singularity of the function represented by the series (W&W §5.4). In this case the nearest singularity is the point $t = 2\pi i$.

(2) W&W §§2.41 and 2.6

(3) W&W §2.53