Roger Apéry's proof that zeta(3) is irrational

Roger Apéry developed a method for searching for continued fraction representations of numbers that have a form such that irrationality criteria can be applied. At a conference in 1978 he outlined a proof that
\[
\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}
\]  

is irrational based on this method. He published a brief note on his results in the journal *Astérisque* in 1979 and a fuller, but still cryptic, account in 1981. Batut and Olivier in 1980 provided a detailed explanation of Apéry's method of generating rapidly converging continued fractions, but other aspects of the irrationality proof did not come within the scope of their paper. Here I attempt to provide a full explanation of Apéry's method of proof.

Apéry's method in outline

We begin with some definitions and facts concerning continued fractions. Suppose \( s_1 \) is some real number that we represent in the following manner
\[
s_1 = \frac{a_1}{b_1 + s_2} \quad ; \quad s_n = \frac{a_n}{b_n + s_{n+1}} \quad (n \geq 2)
\]  

We call this representation of \( s_1 \) a continued fraction. If we define
\[
p_1 = a_1 \quad p_2 = b_2 p_1 \quad p_{n+1} = a_{n+1} p_n + b_{n+1} p_n \quad (n \geq 2)
\]
\[
q_1 = b_1 \quad q_2 = a_2 + b_2 q_1 \quad q_{n+1} = a_{n+1} q_n + b_{n+1} q_n \quad (n \geq 2)
\]  

then it is not hard to show that (Lemma A.1)
\[
s_1 = \frac{p_1}{q_1 + s_2} = \frac{p_n + p_{n-1} s_{n+1}}{q_n + q_{n-1} s_{n+1}} \quad (n \geq 2)
\]  

Conversely, it may be that we commence with a set of real numbers \( a_n, b_n \), with \( p_n, q_n \) defined as in (3). Then if the sequence \( p_n/q_n \) converges to a limit \( s_1 \), we can represent this limit as in (2), since when \( n \geq 1 \) the \( s_n \) are defined by (2).

We call the \( p_n/q_n \) convergents of the continued fraction (whether the sequence converges or not) and the \( p_n \) and \( q_n \) partial numerators and partial denominators respectively.

By way of example, the number \( x = (\sqrt{5} - 1)/2 \) can be expressed in the form (2) with \( s_1 = s_n = x \) and \( a_n = b_n = 1 \). The first of the recurrence relations (3) is that of the Fibonacci sequence \( \{1, 1, 2, ... \} = \{F_n\} \) and the second is that for \( \{F_{n+1}\} \).

Any number that can be expressed as a series can also be expressed quite easily as a continued fraction whose convergents are equal to the partial sums of the series (Lemma A.2).

If \( \lambda_0, \lambda_1, \lambda_2, ... \) is a sequence of non-zero real numbers, \( \lambda_0 \) having the value 1, the transformation
\[
a_k \rightarrow \lambda_k a_k \quad ; \quad b_k \rightarrow \lambda_k b_k \quad ; \quad s_k \rightarrow \lambda_k s_k
\]  

applied to (2) yields another continued fraction with the same value \( s_1 \) (Lemma A.3)

If \( p_{n,0} \) and \( q_{n,0} \) are partial numerators and denominators respectively of a continued fraction and if \( s_{n,k} \) is a double sequence of real numbers \( (k \geq 0) \) and if we construct the infinite rectangular arrays \( p_{n,k} \) and \( q_{n,k} \) iteratively by rows as follows
Apéry's method applied to representations of $\zeta(n)$

To apply the method one must solve two main problems: evaluating \(T_n\) and finding representations of \(s_n\) into the form \(2\) with \(a_n, b_n\) integers.

Clearly \(p_n\) and \(q_n\) are rational if \(a_n\) and \(b_n\) are rational and integers if \(a_n\) and \(b_n\) are integers. If \(s_i\) can be represented as a series with rational coefficients, it is possible using Lemma A.3 to put the continued fraction representation of \(s_i\) into the form \(2\) with \(a_n, b_n\) integers.

The connection between continued fractions and tests for rationality arises as follows. We recall the Theorem (Lemma B) that

\textit{s} is irrational if there exist integers h and k such that \(0 < |ks_i - h| < \varepsilon\) for any given \(\varepsilon > 0\)

Re-arranging \((4)\) and using \((3)\) we obtain

\[
q_n s_1 - p_n = \frac{(q_n p_{n-1} - p_n q_{n-1}) s_{n+1}}{q_n + q_{n-1} s_{n+1}} = (-1)^{n-1}(a_{n-1} a_{n-2} \ldots a_1) s_{n+1} = T_n \quad \text{............ \((6)\)}
\]

If \(\lim_{n \to \infty} \frac{p_n}{q_n} = s_i\) we also have

\[
s_i = \frac{p_n}{q_n} + \lim_{k \to \infty} \sum_{j=n+1}^{k} \left[ \frac{p_j}{q_j} - \frac{p_{j-1}}{q_{j-1}} \right]
\]

and so

\[
q_n s_1 - p_n = q_n \sum_{j=n+1}^{\infty} \left[ \frac{p_j}{q_j} - \frac{p_{j-1}}{q_{j-1}} \right] = q_n \sum_{j=n+1}^{\infty} \frac{(-1)^j a_j a_{j-1} \ldots a_1}{q_j q_{j-1}} = T_n \quad \text{............ \((6a)\)}
\]

If we can find a representation of \(s_i\) such that \(q_n\) and \(p_n\) are integers with common divisor \(d\) and \(0 < |T_n|/d \leq \varepsilon\) for any given \(0 < \varepsilon < 1\) when \(n\) is sufficiently large, then the theorem applies and \(s_i\) is irrational.

Suppose that for \(n > 1\) \(s_n\) can be expressed as

\[
s_n = \sigma_n + \varepsilon_n\quad \text{.................................................. \((7)\)}
\]

where \(\sigma_n\) is a polynomial in \(n\) with integer coefficients and \(\varepsilon_n = O(1/n)\) (that is, for all \(n\) greater than some specific value, \(|\varepsilon_n| < A/n\) where \(A\) is a positive real number independent of \(n\)).

Equation \((4)\) suggests that the transformation \((5)\) might yield convergents having the same limit as the original continued fraction and in fact for \(\zeta(3)\) and a variety of more general cases this is indeed so. In the case of \(\zeta(3)\) the partial numerators and denominators \(p_{n,k}\) and \(q_{n,k}\) respectively have large common factors and in particular the diagonal elements \(p_{n,n}\) and \(q_{n,n}\) yield a version of \((6a)\) to which the theorem can be applied.

To apply the method one must solve two main problems: evaluating \(\sigma_n\) and finding representations of \(p_{n,n}\) and \(q_{n,n}\) that enable us to identify the common factors.

\textbf{Apéry's method applied to} \(\zeta(3)\)

Starting with \((1)\) then by virtue of Lemmas A.2 and A.3 (with \(\lambda_1 = 1\) and \(\lambda_n = n^3\)) we obtain for \(s_i = \zeta(3)\) a continued fraction with integer parameters (Lemma C.1):

\[
a_i = b_i = 1 \quad ; \quad a_n = -(n-1)^6 \quad b_n = n^3 + (n-1)^3 \quad (n \geq 2)
\]

\[
s_{n+1} = -n^3 + 2n^2 - 2n + 1 - \frac{1}{6n^2} - \frac{1}{6n^3} + O\left(\frac{1}{n^4}\right) = \sigma_{n+1} + O\left(\frac{1}{n^4}\right)
\]

\[
q_n = (n!)^3 \quad ; \quad p_n = (n!)^3 \sum_{k=1}^{n} \frac{1}{k^3} = q_n \cdot \sigma_n\quad \text{.................................................. \((8)\)}
\]
This tells us nothing more than that
\[
\zeta(3) - \Sigma_n = \sum_{k=n+1}^{\infty} \frac{1}{n^3} = P_{n+1}
\]
with (Lemma C.1) \(P_{n+1} = O(1/n^2)\)

If \(L_n\) is the smallest positive integer divisible by all of the integers 1, 2, ... , \(n\) then \((L_n)^3\Sigma_n\) is an integer. As a consequence of Chebyshev's investigations in the mid 19th century of the distribution of prime numbers we know (e.g. Hardy and Wright Theorem 414) that
\[
L_n = O(e^n)
\]
and so
\[
|q_n s_1 - p_n| = |(L_n)^3 \zeta(3) - (L_n)^3 \Sigma_n|
\]
increases with \(e^n n^{-2}\) and the Theorem cannot be applied.

In the hope of improving on this situation, we construct a hierarchy of continued fractions \(s_{n,1}, s_{n,2}\) and so on using the method described by (5). We commence with
\[
p_n = 0; \quad q_n = q_n \quad \text{and} \quad s_n = s_n \quad \text{as in (8).}
\]
In Lemma C.2 we show that we obtain in consequence a set of continued fractions
\[
s_{n,k} = \frac{a_{n,k}}{b_{n,k} + s_{n+1,k}}
\]
where
\[
a_{1,k} = p_{1,k} = (k!)^3(1 + (2k^2 + 2k + 1) \Sigma_k) \quad b_{1,k} = q_{1,k} = (k!)^3(2k^2 + 2k + 1)
\]
\[
a_{n,k} = -(n-1)^6 \quad b_{n,k} = n^5 + (n-1)^5 + 2k(k+1)(2n-1) \quad (n > 1)
\]
\[
s_{n+1,k} = \sigma_{n+1,k} + \epsilon_{n+1,k} = -n^3 + 2(k + 1)n^2 - 2(k + 1)^2 n + (k + 1)^3 + O(\frac{1}{n^2})
\]
and partial numerators and partial denominators of \(s_{1,k}\) are
\[
p_{n,k} = (n!)^3 \left( \pi_{n,k}^{(1)} + \pi_{n,k}^{(2)} \Sigma_n \right)
\]
\[
q_{n,k} = (n!)^3 \pi_{n,k}^{(2)}
\]
where \(\pi_{n,k}^{(1)}\) and \(\pi_{n,k}^{(2)}\) are polynomials with integer coefficients of degree \(2(k - 1)\) (at most) and \(2k\) respectively in \(n\) and consequently
\[
s_{1,k} = \lim_{n \to \infty} \frac{p_{n,k}}{q_{n,k}} = \lim_{n \to \infty} \Sigma_n = \zeta(3)
\]

Table 5.3 of Batut and Olivier gives the values of \(p_{n,k}\) and \(q_{n,k}\) for small \(n,k\). These values are invariant with respect to exchange of \(n\) and \(k\) and \(q_{n,k}\) is divisible by \((n!)^3(k!)^3\). The symmetry property can be shown to be generally true by Lemma A.4(ii), which shows that \(a_{k,n} = a_{n,k}\) and \(b_{k,n} = b_{n,k}\). We can (Lemma C.3) solve the recurrences (5) to find the 'closed form' expressions (10). In these expressions \((r)\) - the 'shifted factorial' - is the product of \(s\) consecutive positive integers, the largest of which is \(r\). \((r)\) is divisible by \(s!\) (see Lemma C.3 - Appendix) and consequently \((s) = (r) = s\). Thus the expressions (10) not only verify the divisibility property of \(q_{n,k}\) but also show that \((L_n)^3p_{n,k}\) has the same property.

\[
\pi_{n,k}^{(2)} = (k!)^3 \mu_{n,k} = (k!)^3 \sum_{j=0}^k \frac{(n + j)_{2j}(k + j)_{2j}}{(j!)^4}
\]
\[
\pi_{n,k}^{(1)} = (k!)^3 \nu_{n,k} = (k!)^3 \sum_{j=0}^{k-1} \frac{(n + j)_{2j}(k + j)_{2j}}{(j!)^4} [\Sigma_k - \Sigma_j] \quad (k \leq n, \Sigma_0 \text{ defined as 0})
\]
These results take us closer to an expression of the form (6) to which we can apply the
Theorem, since
\[ |q_{n,k} \zeta(3) - p_{n,k}| = \frac{(n-1)!}{|q_{n,k} + \lambda_{n-1,k} s_{n+1,k}|} \frac{(n-1)!}{s_{n+1,k}} = \frac{(n-1)!}{|q_{n-1,k+1} + \lambda_{n-1,k} O\left(\frac{1}{n^2}\right)|} \frac{(n-1)!}{s_{n+1,k}} \]

Since \( p_{n,k} \) and \( q_{n,k} \) are divisible by \( d = (n!)^3/(L_n)^3 \) and \( s_{n+1,k} \neq 0 \) the result is that
\[ 0 < \left| \frac{p_{n,k}}{d} \right| = \frac{L_n^3}{O(n^{2k+5})} \]
The right hand side increases as \( e^{3n^{2k-5}} \) (since \( k \) must be taken to be constant as \( n \) increases). The result suggests however that if we were to use the diagonal elements \( p_{n,n} \) and \( q_{n,n} \) (which, by Lemma A.4(iii) are partial numerators and partial denominators respectively of a continued fraction) it might be possible to obtain an expression of the form
\[ 0 < \left| \frac{p_{n,n}}{d} \right| = \frac{L_n^3}{O(n^{2n+2})} = O(e^{3n-(2n+5)\ln n}) \]
which has the desired properties since it becomes arbitrarily small as \( n \) increases. The result we obtain will not be as strong as this, but still sufficient to show that \( \zeta(3) \) is irrational.

By Lemma A.4(iii), the continued fraction formed by the diagonal has
\[ a_1 = 6 \; ; \; b_1 = 5 \; ; \; a_n = -n^3(n - 1)^3 \; ; \; b_n = n^3(34n^3 - 51n^2 + 27n - 5) \quad (n > 1) \]
and \( p_{n,n} \) and \( q_{n,n} \) as defined in (9).

If we apply Lemma A.3 with \( \lambda_0 = 1, \lambda_n = 1/n^6 \) (\( n > 0 \)) we obtain a continued fraction with
\[ a_1 = 6 \; ; \; b_1 = 5 \; ; \; a_n = -(n - 1)^3/n^3 \; ; \; b_n = (34n^3 - 51n^2 + 27n - 5)/n^3 \quad (n > 1) \]
\[ q_n = \mu_{n,n} = \sum_{j=0}^{n} \left[ \frac{(n+j)_3}{(j!)^3} \right]^2 \]
\[ p_n = \nu_{n,n} + \mu_{n,n} \Sigma_n \quad \text{with} \quad \nu_{n,n} = \sum_{j=0}^{n-1} \left[ \frac{(n+j)_3}{(j!)^3} \right]^2 \left[ \Sigma_n - \Sigma_j \right] \quad \text{......................... (11)} \]

This continued fraction has the limit \( \zeta(3) \) (Lemma C.4) and it is seen from (6a) since
\[ a_1, \ldots, a_j = (-1)^{n-1} \frac{6}{n^3} \]
that
\[ 0 < |(L_n)^3 q_n \zeta(3) - (L_n)^3 p_n| = (L_n)^3 q_n \sum_{j=n+1}^{\infty} \frac{6}{k^3 q_j q_{j-1}} = O\left(\frac{e^{3n}}{n^3 q_n}\right) \]

It is not difficult to show (Lemma C.5) that \( q_n \) is an increasing function of \( n \) and that for some positive integer \( A \) independent of \( n \), \( q_n > Ae^{\ln 33} > Ae^{3n} \). This is sufficient to prove the result.
Lemmas: A (Theorems about continued fractions generally)

Lemma A.1: Let \( s_1 \) be some real number that we can represent in the following manner:

\[
s_1 = \frac{a_1}{b_1 + s_2}; \quad s_n = \frac{a_n}{b_n + s_{n+1}} \quad (n \geq 2)
\]

If we define

\[
p_1 = a_1; \quad p_2 = b_2 p_1; \quad p_{n+1} = a_{n+1} p_{n+1} + b_{n+1} p_n \quad (n \geq 2)
\]

\[
q_1 = b_1; \quad q_2 = a_2 + b_2 q_1; \quad q_{n+1} = a_{n+1} q_{n+1} + b_{n+1} q_n \quad (n \geq 2)
\]

then

\[
s_1 = \frac{p_1}{q_1 + s_2} = \frac{p_n + p_{n-1} s_{n+1}}{q_n + q_{n-1} s_{n+1}} \quad (n \geq 2)
\]

Proof: When \( n = 1 \) and \( 2 \), the Lemma is clearly correct. Substituting

\[
s_{n+1} = \frac{a_{n+1}}{b_{n+1} + s_{n+2}}
\]

in the expression

\[
s_1 = \frac{p_n + p_{n-1} s_{n+1}}{q_n + q_{n-1} s_{n+1}}
\]

we easily see that if (c) is true for the integers 1, 2, ... \( n+1 \), it is also true for \( n+2 \).

Lemma A.2: Let \( s_1 = \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{n} u_k + P_{n+1} \) be convergent. Define

\[
u_0 = 1; \quad s_1 = P_1; \quad s_{k+1} = -\frac{P_{k+1}}{P_k} = -\left(1 - \frac{u_k}{P_k}\right) \quad (k \geq 1)
\]

Then \( s_1 \) can be expressed in the form

\[
s_1 = \frac{u_1}{1 + s_2}; \quad s_k = \frac{u_k}{1 + \frac{u_k}{u_{k-1} + s_{k+1}}} \quad (2 \leq k)
\]

and the partial numerator and partial denominator of this continued fraction are

\[
p_n = \sum_{k=1}^{n} u_n \quad \text{and} \quad q_n = 1
\]

Proof: If \( k = 1 \) then

\[
\frac{s_2}{s_1} = \frac{-\left(s_1 - u_1\right)}{s_1} \quad \text{and so} \quad s_1 = \frac{u_1}{1 + s_2}
\]

If \( k \geq 2 \) then since \( P_{k+1} \cdot R_k = -u_k \) we have

\[
\frac{P_{k+1}}{P_k} = -\frac{u_k}{P_k}
\]

Similarly

\[
1 - \frac{P_k}{P_{k-1}} = -\frac{u_{k-1}}{P_k}
\]

and so

\[
u_{k-1} \frac{P_{k+1}}{P_k} - u_{k-1} = u_k - \frac{u_k}{\left(\frac{P_k}{P_{k-1}}\right)}
\]
This re-arranges to give
\[ \frac{-P_k - P_{k-1}}{1 + \frac{u_k}{u_{k-1}} - \frac{P_{k+1}}{P_k}} = \frac{u_k}{u_{k-1}}. \]

The second part of the Lemma is easily proved by induction using the recurrence formula for \( p_n \) and \( q_n \).

**Lemma A.3:** Let \( \lambda_0 = 1, \lambda_1, \lambda_2, \ldots \) be a sequence of non-zero real numbers. We define \( s_1 \) as in Lemma 1. Then \( s_1 \) remains unchanged if we substitute the following primed quantities for the respective unprimed quantities in equations (a), (b), (c) of Lemma 1.

- \( a_k' = \lambda_{k-1} \lambda_k a_k \)
- \( b_k' = \lambda_k b_k \)
- \( s_k' = \lambda_{k-1} s_k \quad (k \geq 2) \)
- \( p_k' = \lambda_1 \lambda_2 \ldots \lambda_k p_k \)
- \( q_k' = \lambda_1 \lambda_2 \ldots \lambda_k q_k \)

**Proof:** \( s_1 \) as defined in (a) is unchanged since
\[ \lambda_{k-1} s_k = \frac{\lambda_{k-1} \lambda_k a_k}{\lambda_k b_k + \lambda_k s_{k+1}} \quad (k \geq 1) \]

It is true with respect to (c) for \( k = 1 \) and \( k = 2 \). Suppose it is true for all \( k \leq n \). Then since
\[ (\lambda_{n+1} a_{n+1} + (\lambda_{n+1} b_{n+1})(\lambda_{n+1} \ldots \lambda_1 p_{n+1}) = (\lambda_{n+1} \ldots \lambda_1) (a_{n+1} p_{n+1} + b_{n+1} p_{n+1}) = \]

it is true for \( k = n + 1 \). A similar result applies for \( q_{n+1} \).

**Lemma A.4:** Let \( a_n \) and \( b_n \) (\( n \geq 1 \)) be sequences of real numbers and let \( u_n = u_{n,0} \) be a sequence defined as in (3). Let \( \sigma_{nk} \) be a double sequence of real numbers (\( k \geq 0 \)). If we construct a double sequence iteratively by rows as follows:

- \( u_{n+1,k} = u_{n+1,k} + \sigma_{n+2,k} u_{n,k} \)

then

(i) the elements in the rows follow the recurrence
\[ u_{n,k} = a_{n,k} u_{n-1,k} + b_{n,k} u_{n-2,k} \quad (n \geq 2, k \geq 0) \]

(ii) the elements in the columns follow the recurrence
\[ u_{n,k} = d_{n,k} u_{n-2,k-2} + (\sigma_{n+2,k-1} + R_{n+1,k-1}) u_{n,k-1} \quad (k \geq 2) \]

(iii) the diagonal elements follow the recurrence
\[ u_{n,n} = a'_n u_{n-2,n-2} + b'_n u_{n-1,n-1} \quad (n \geq 2) \]

where \( a_{nk} \) and \( b_{nk} \) are defined recursively by
\[ a_{n,k+1} = a_{n,k} d_{n-1,k} \]
\[ b_{n,k+1} = R_{n,k} - \sigma_{n,k} d_{n-1,k} \quad (n \geq 2, k \geq 0) \]

with \( R_{nk} \) and \( d_{nk} \) defined recursively by
\[ R_{n,k} = \sigma_{n+2,k} + b_{n+1,k} d_{n,k} \]
\[ d_{n,k} = a_{n+1,k} - \sigma_{n+1,k} R_{n,k} \]

and \( a'_n \) and \( b'_n \) are defined recursively by

...
We refer to the following schema

\[ a_n' = \frac{R\n,n-1}{R\n-1,n-2} d_{n,n-2} a_{n,n-2} = \frac{R\n,n-1}{R\n-1,n-2} d_{n-1,n-2} a_{n,n-1} \]

\[ b_n' = R\n,n-1 \left( R\n,n-2 + \frac{d_{n-2}}{R\n-1,n-2} \right) + a_{n+1,n} \]

\[ = d_{n,n-1} + R\n,n-1 \left( R\n-1,n-1 + \frac{a_{n,n-1}}{R\n-1,n-2} \right) \]

Proof:

(i) The proposition is true for all \( n \geq 2 \) if \( k = 0 \). If \( k > 0 \) we have

\[ u_{n,k+1} = u_{n+1,k} + \sigma_{n+2,k} u_{n,k} = a_{n+1,k} R_n n_{-1,k} u_{n-1,k} + (a_{n+1,k} + b_{n,k} R_n n_{k} u_{n-1,k}) \]

\[ u_{n-2,k+1} = u_{n-1,k} + \sigma_{n+1,k} u_{n-2,k} \]

\[ u_{n-1,k+1} = u_{n,k} + \sigma_{n+1,k} u_{n-1,k} = a_{n,k} u_{n-2,k} + R_n n_{-1,k} u_{n-2,k} \]

\[ a_{n,k+1} u_{n-2,k+1} + b_{n,k+1} u_{n-1,k+1} = (a_{n,k+1} \sigma_{n,k} + a_{n,k} b_{n,k+1}) u_{n-2,k} + u_{n-1,k} \]

\[ + (a_{n,k+1} + b_{n,k} R_n n_{-1,k}) u_{n-1,k} \]

\[ a_{n,k} = a_{n,k} R_n n_{k} u_{n-2,k} + \frac{d_{n,k}}{d_{n-1,k}} (a_{n,k} - \sigma_{n,k} R_n n_{-1,k}) + R_n n_{k} R_n n_{-1,k} u_{n-1,k} \]

\[ = a_{n,k} R_n n_{k} u_{n-2,k} + (a_{n+1,k} + b_{n,k} R_n n_{k}) u_{n-1,k} \]

\[ = u_{n+1,k+1} \]

(ii) The array \( u_{n,k} \) is populated by columns via the expression

\[ u_{n,k} = u_{n+1,k-1} + \sigma_{n+2,k-1} u_{n,k-1} \]

thus:

\[ U_{n,k-2} \quad U_{n+1,k-2} \quad U_{n+2,k-2} \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ U_{n,k-1} \quad U_{n+1,k-1} \]

\[ \downarrow \]

\[ U_{n,k} \]

We need to express \( u_{n+1,k+1} \) in terms of \( u_{n,k-1} \) and \( u_{n,k} \)

\[ u_{n+1,k-1} = u_{n+2,k-2} + \sigma_{n+3,k-2} u_{n+1,k-2} \]

\[ = a_{n+2,k-2} u_{n,k-2} + b_{n+2,k-2} u_{n+1,k-2} + \sigma_{n+2,k-2} (u_{n,k-1} - \sigma_{n+2,k-2} u_{n,k-2}) \]

\[ = (a_{n+2,k-2} - \sigma_{n+3,k-2} \sigma_{n+2,k-2}) u_{n,k-2} + \sigma_{n+3,k-2} u_{n,k-1} + b_{n+2,k-2} (u_{n,k-1} - \sigma_{n+2,k-2} u_{n,k-2}) \]

\[ = (a_{n+2,k-2} - \sigma_{n+2,k-2} R_n n_{-1,k-2}) u_{n,k-2} + \sigma_{n+1,k} u_{n,k-1} + b_{n+2,k-2} (u_{n,k-1} - \sigma_{n+2,k-2} u_{n,k-2}) \]

\[ = d_{n+1,k-2} u_{n,k-2} + R_n n_{1,k-1} u_{n,k-1} \]

So \( u_{n,k} = d_{n+1,k-2} u_{n,k-2} + (\sigma_{n+2,k-1} + R_n n_{1,k-2}) u_{n,k-1} \)

(iii) We first note that the following recurrences apply

\[ u_{n,k} = u_{n+1,k-1} + \sigma_{n+2,k-1} u_{n,k-1} = a_{n+1,k-1} u_{n-1,k-1} + R_n n_{k-1} u_{n,k-1} \]

\[ = a_{n+1,k-1} u_{n-1,k-1} + R_n n_{k-1} (u_{n-1,k-1} - \sigma_{n,k}) u_{n-1,k-1} = d_{n,k-1} u_{n-1,k-1} + R_n n_{k-1} u_{n-1,k} \]

We refer to the following schema
By (i) and the preceding recurrences, there exist relationships
\[ u_{n,n} = a_{n+1,n-1} u_{n-1,n-1} + R_{n,n-1} u_{n,n-1} \]
\[ u_{n-1,n} = d_{n-1,n-2} u_{n-2,n-2} + R_{n-1,n-2} u_{n-2,n-1} \]
\[ u_{n,n-1} = a_{n,n-1} u_{n-2,n-1} + b_{n,n-1} u_{n-1,n-1} \]
and from these we easily find
\[ u_{n,n} = -\frac{R_{n,n-1}}{R_{n-1,n-2}} d_{n-1,n-2} a_{n,n-1} u_{n-2,n-2} \]
\[ + (a_{n+1,n-1} + R_{n,n-1} \left( b_{n,n-1} + \frac{a_{n,n-1}}{R_{n-1,n-2}} \right) ) u_{n-1,n-1} \]
and since
\[ R_{n,p} b_{n,p-1} = R_{n,p} R_{n-1,p-1} - R_{n,p} \sigma_{n+1,p-1} = R_{n,p} R_{n-1,p-1} + d_{n,p-1} - a_{n+1,p-1} \]
the first half of (iii) follows.
Now referring to the following schema

we have, using (ii) and the recurrences at the head of this section
\[ u_{n,n} = d_{n,n-1} u_{n-1,n-1} + R_{n,n-1} u_{n-1,n} \]
\[ u_{n-1,n} = d_{n-1,n-2} u_{n-2,n-2} + R_{n-1,n-2} u_{n-2,n-1} \]
\[ u_{n,n-1} = d_{n,n-2} u_{n-1,n-2} + (\sigma_{n+1,n-1} + R_{n,n-2}) u_{n-1,n-1} \]
from which we easily find
\[ u_{n,n} = -\frac{R_{n,n-1}}{R_{n-1,n-2}} d_{n,n-2} a_{n,n-2} u_{n-2,n-2} + \left( R_{n,n-1} \left( R_{n,n-2} + \frac{d_{n,n-2}}{R_{n-1,n-2}} \right) + a_{n+1,n-1} \right) u_{n-1,n-1} \]

**Lemma B: Theorem: criterion for irrationality**

**Theorem:** Let \( x \) be a real number. If for any given real number \( 1 > \varepsilon > 0 \), there exist integers \( h \) and \( k \) such that, \( 0 < |kx - h| < \varepsilon \), then \( x \) is irrational.

**Proof:** since \( |kx - h| \neq 0 \), \( h/k \neq x \). Since \( \varepsilon < 1 \) then \( k \neq 0 \). The rest follows by the counterpositive to the following proposition:

if \( x \) is rational then there exists a positive integer \( n \) such that for every pair of integers \( h, k \) where \( k \neq 0 \) and \( h/k \neq x \), \( |kx - h| \geq 1/n \), with \( n \) being independent of \( h, k \)
The proposition is true if \( x = 0 \) because then \(|kx - h| = |h| \geq 1\). Otherwise suppose \( x = \frac{a}{b} \) with \( a, b \neq 0 \) and \( \text{g.c.d.}(a, b) = 1 \). Then \(|kx - h| = |ka - hb|/|b|\). The numerator is an integer and cannot be zero because if it were, we would have \( ka - hb = 0 \) and \( h/k = a/b = x \). Therefore the numerator is \( \geq 1 \). Identifying \(|b|\) (which is uniquely determined by \( x \)) with \( n \) we have the result.

**Lemmas C: Lemmas specific to the case of \( \zeta(3) \)**

**Lemma C.1:** \( \zeta(3) \) can be represented by the continued fraction (8)

Proof: By virtue of Lemma A.2 we obtain from the series (1) a continued fraction for \( \zeta(3) \) with

\[
a_1 = b_1 = 1 ; \quad a_n = \frac{-(n-1)^3}{n} ; \quad b_n = 1 - a_n \quad (n > 1)
\]

Starting with (1) then by virtue of Lemmas A.2 and A.3 (with \( \lambda_1 = 1 \) and \( \lambda_n = n^3 \)) we obtain for \( s_1 = \zeta(3) \) a continued fraction with integer parameters:

\[
a_1 = b_1 = 1 ; \quad a_n = -(n-1)^6 ; \quad b_n = n^3 + (n-1)^3 \quad (n \geq 2)
\]

and

\[
s_{n+1} = -n^3 + \frac{1}{P_n}
\]

with

\[
P_{n+1} = \zeta(3) - \sum_{k=1}^{n} \frac{1}{k^3} = \zeta(3) - \Sigma_n
\]

and the partial numerators and partial denominators are

\[
q_n = (n!)^3 ; \quad p_n = (n!)^3 \Sigma_n = q_0 \Sigma_n
\]

From the Euler-Maclaurin sum formula (as stated at §7.21 of Whittaker and Watson) we have

\[
\sum_{k=n}^{\infty} \frac{1}{k^3} = P_n = \int_{n}^{\infty} \frac{1}{x^3} \, dx + \frac{1}{2n} + \sum_{m=1}^{\infty} \frac{(-1)^m B_m}{(2m)!} \left[ \frac{1}{n^3} \right]^{(2m-1)}
\]

where \( \left[ \frac{1}{n^3} \right]^{(2m-1)} \) denotes the 2m-1 th derivative of the quantity in the square brackets and the \( B_m \) are the Bernoullian numbers as defined by W&W with \( B_1 = 1/6, B_2 = 1/30, B_3 = 1/42 \) and so on. Thus

\[
P_n = \frac{1}{2n^2} + \frac{1}{2n^3} + \frac{1}{4n^4} - \frac{1}{12n^6} + \frac{1}{12n^8} + O\left(\frac{1}{n^{10}}\right)
\]

Using long division to obtain \( 1/P_n \) from \( P_n \) we find

\[
s_{n+1,0} = -n^3 + 2n^2 - 2n + 1 - \frac{1}{6n^2} - \frac{1}{6n^3} + O\left(\frac{1}{n^4}\right) = \sigma_{n+1,0} + O\left(\frac{1}{n^3}\right)
\]

**Lemma C.2:** Commencing with \( p_{n,0} \) and \( q_{n,0} \) as defined in (9) and \( s_{n,0} \) as defined in (11) and using (5) we can construct by rows a hierarchy of continued fractions having the following properties:

\[
a_{1,k} = (k!)^3 (1 + (2k^2 + 2k + 1) \Sigma_n) ; \quad b_{1,k} = (k!)^3 (2k^2 + 2k + 1) \quad \ldots \quad (a)
\]

\[
a_{n,k} = -(n-1)^6 \quad b_{n,k} = n^3 + (n-1)^3 + 2k(k+1)(2n-1) \quad (n > 1) \quad \ldots \quad (b)
\]

and whose partial numerators and partial denominators are

\[
p_{n,k} = (n!)^3 \left( \pi_{n,k}^{(1)} + \pi_{n,k}^{(2)} \Sigma_n \right)
\]
\[ q_{n,k} = (n!)^3 \pi_{n,k}^{(2)} \tag{c} \]

where \( \pi_{n,k}^{(1)} \) and \( \pi_{n,k}^{(2)} \) are polynomials of degree \( 2(k - 1) \) (at most) and \( 2k \) respectively in \( n \) and also

\[ s_{n+1,k} = -n^3 + 2(k+1)n^2 - (k+1)^2n + (k+1)^3 + O\left(\frac{1}{n^2}\right) \tag{d} \]

**Proof:** By induction. When \( k = 1 \) we readily obtain:

\[ p_{n,1} = (n!)^3 \left[ \pi_{n,1}^{(1)} + \pi_{n,1}^{(2)} \right] \]

\[ q_{n,1} = (n!)^3 \pi_{n,1}^{(2)} \]

where

\[ \pi_{n,1}^{(2)} = 2n^2 + 2n + 1 \quad \text{and} \quad \pi_{n,1}^{(1)} = 1 \]

This establishes (c). Since \( a_{1,1} = p_{1,1} \) and \( b_{1,1} = q_{1,1} \) it also establishes (a). (b) is established by applying Lemma A.5(i).

From (4) we have:

\[ s_{n+1,1} = \frac{p_{n,1} - q_{n,1} \zeta(3)}{p_{n-1,1} - q_{n-1,1} \zeta(3)} = \frac{p_{n,1} - q_{n,1} \zeta(3) + \sigma_{n+1,0}}{p_{n,1} - q_{n,1} \zeta(3) + \sigma_{n+1,0}} \]

\[ = \frac{-s_{n+2,0} \left( p_{n,1} - q_{n,1} \zeta(3) \right) + \sigma_{n+2,0}}{p_{n,0} - q_{n,0} \zeta(3)} \]

\[ = s_{n+2,0} - \frac{\sigma_{n+2,0}}{s_{n+1,0}} \]

By (10), the numerator has the form

\[ \frac{a}{n^3} + \frac{b}{n^4} + O\left(\frac{1}{n^4}\right) \]

and using the binomial series for \((1 + 1/n)^k\), the denominator has the form

\[ \frac{a}{n^3} + \frac{b - 2a}{n^4} + O\left(\frac{1}{n^4}\right) \]

Thus, carrying our long division, we find

\[ s_{n+1,1} = s_{n+1,0} \left( 1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right) \right) = -n^3 + 4n^2 + O(n) \]

Further terms can be found through the following process. By (2) and Lemma A.5.1 we have

\[ s_{n+1,1} = \frac{a_{n+1,1}}{b_{n+1,1} + s_{n+2,1}} = \frac{-n^6}{n^3 + (n+1)^3 + 4(2n+1) + s_{n+2,1}} \]

If we let

\[ s_{n+1,1} = -n^3 + 4n^2 + cn + d + \frac{e}{n} + \frac{f}{n^2} + \frac{g}{n^3} + O\left(\frac{1}{n^4}\right) \]

(where \( c, \ldots, g \) are independent of \( n \)), cross-multiply then equate coefficients of \( n \) we find:

- \( 7c = -56 \) and so \( c = -8 \)
- \( 8d + c^2 + 20c = -32 \) and so \( d = 8 \)
- \( 9e + c^2 + 8e + 16d + 2cd = 0 \) so \( e = 0 \)
- \( 10f + 8d + 11e + d^2 + cd + 2ec = 0 \) and so \( f = -32/5 \)

hence

\[ s_{n+1,1} = -n^3 + 4n^2 - 8n + 8 + \frac{32}{5n^2} + O\left(\frac{1}{n^3}\right) \]
where c, ..., g are independent of n, and find

As before, we take

obtain a quadratic diophantine equation with two integer solutions.) By (2) and Lemma A.5

This proves that (c) is true for k+1.

It is not hard to see from analysis of its components that

Let

This establishes the truth of (a) for k + 1. We establish the truth of (b) since, with reference to

When k > 2 we have from Lemma A.5(ii)

This establishes the truth of (a) for k + 1. We establish the truth of (b) since, with reference to

for the row k + 1 using eq’n (9). It is not hard to show that

variable up to some fixed value k. We form the partial numerators and partial denominators

This establishes the four propositions for k = 1

4(k + 2)d + c

(4k+7)c = -2(k + 2)

\[ \begin{align*}
4(k + 2)d + c^2 + 2(k + 2)(2k + 5)c + 4(k + 2)^2 &= 0 \\
\text{and so } d &= (k + 2)^2
\end{align*} \]
It is not hard to see that
\[(4k + 5)e + 4(k + 2)2d + 2cd + 2(k + 2)^2c + c^2 = 0 \text{ so } e = 0\]
\[(4k + 10)f + 2(k + 2)^2d + d^2 + cd + \text{terms with } e \text{ as a factor} = 0 \text{ so } f = -(k+2)^2/(4k + 10)\]

Lemma C.3: if \(\pi^{(2)}_{n,k}\) and \(\pi^{(1)}_{n,k}\) are as defined in (9) then they have closed form representations as described in (10).

Proof: first we deal with \(\pi^{(2)}_{n,k}\). We write
\[\pi^{(2)}_{n,k} = (k!)^3 \mu_{n,k}\]
from which, using the recurrence (5), we obtain
\[(k+1)^3 \mu_{n,k+1} = (n+1)^3 \mu_{n+1,k} + \sigma_{n+2,k} \mu_{n,k}\] .............................. (a)
with \(\mu_{n,0} = 1\).

Rather than using a polynomial form (linear combination of powers of \(n\)) we use a shifted factorial form (linear combination of shifted factorials in \(n\)) to define \(\mu\). Shifted factorials and some properties are described in the Appendix to the Lemma. The rationale for using them is that the recurrence (a) is a form of partial difference equation. Since the difference relation
\[(n+1)\mu_{n+1,k} - n\mu_{n,k} = f_{n+1,k}\]
is the discrete analogue of the differential \(d^2 x = n^2 x\), this suggests that it may be fruitful to search for solutions of difference equations expressed in shifted factorial form.

Using (a), and making use of the identities in the Appendix, we can calculate \(\mu_{n,k}\) for
\[k = 1, 2, \ldots \text{ and these suggest the general formula, when } k \leq n\]
\[\mu_{n,k} = \sum_{j=0}^{k} c_j^{(k)}(n + j)_{2j} \quad \text{with} \quad c_j^{(k)} = \frac{(k+j)_{2j}}{(j!)^4} \] .............................. (b)

We prove this by induction on \(k\). The proposition is true for all \(n\) when \(k = 0, 1\). We suppose it is true for all \(n\) for each value of the variable \(k\) less than or equal to some fixed value \(k\).

We can easily transform (b) to the following
\[(k+1)^3 \mu_{n,k+1} = (n+1)^3(\mu_{n+1,k} - \mu_{n,k}) + 2(n+1)^2(k+1) - 2(n+1)(k+1)^2 + (k+1)^3 \mu_{n,k}\] .............................. (c)

On the LHS, we note that if \(j \leq n\) any polynomial of degree \(2k\) in \(n\) can be written
\[P_{n,k} = \sum_{j=0}^{k} c_j^{(k)}(n + j)_{2j} + \sum_{j=1}^{k} d_j^{(k)}(n + j)_{2j-1}\]
where the \(c_j^{(k)}\) and \(d_j^{(k)}\) are unique. So we can write
\[\mu_{n,k+1} = \sum_{j=1}^{k} c_j^{(k+1)}(n + j)_{2j} + \sum_{j=1}^{k+1} d_j^{(k+1)}(n + j)_{2j-1} + c_0^{(k+1)} \] .............................. (d)

If we substitute (b) into the right hand side of (c) we obtain an expression of the form
\[(k+1)^3 \mu_{n,k+1} = \sum_{j=0}^{k} c_j^{(k)} A_j^{(k)}(n) \] .............................. (e)

It is not hard to see that
\[A_j^{(k)}(n) = 2(k+1)(n+1)_{2j} - 2k(k+1)(n+1)_{2j-1} + (k+1)^3\]
To evaluate \(A_j^{(k)}\) for \(j > 0\) we can make use of the identities in the Appendix and find
\[A_j^{(k)}(n) = (2j+2k+2)(n+j+1)_{2j+2} + 2j(j+1) - 2k(k+1)(n+j+1)_{2j+1}\]
\[+ [2j^3 + 2j^2 + 2j(k+1)^2 + (k+1)^3](n+j)_{2j} + 2j^4(n+j)_{2j-1}\]
and
\[(k+1)^3 \mu_{n,k+1} = \sum_{j=1}^{k} B_j^{(k)} c_j^{(k)} (n+j+1)_{2(j+1)} + D_j^{(k)} c_j^{(k)} (n+j+1)_{2(j+1)-1} + \sum_{j=1}^{k} C_j^{(k)} c_j^{(k)} (n+j)_{2j} + E_j^{(k)} c_j^{(k)} (n+j)_{2j-1} + c_0^{(k)} A_0^{(k)} \]

\[= 2(2k+1) c_k^{(k)} (n+k+1)_{2(k+1)} + \sum_{j=1}^{k} [B_j^{(k)} c_j^{(k)} + C_j^{(k)} c_j^{(k)}] (n+j)_{2j} + [D_j^{(k)} c_j^{(k)} + E_j^{(k)} c_j^{(k)}] (n+j)_{2j-1} + c_0^{(k)} (k+1)^3 \]

\[\text{................................. (f)}\]

where

\[B_j^{(k)} = 2(j+k)\]
\[C_j^{(k)} = 2j^3 + 2j^2 (k+1) + 2j (k+1)^2 + (k+1)^3 = 2(j^2 + (k+1)^2)(j+k+1) - (k+1)^3\]
\[D_j^{(k)} = 2(j(j-1) - k(k+1)) = -2(k+j)(k-j+1)\]
\[E_j^{(k)} = 2j^4\]

We see at once that
\[c_0^{(k+1)} = c_0^{(k)} = 1\]

Now if we compare the coefficients of \((n+k+1)_{2k+2}\) on the RHS of (d) and (f) we have
\[c_{k+1}^{(k+1)} = \frac{(2k+2)(2k+1)}{(k+1)^4} c_k^{(k)} = \frac{2(k+1)!}{(k+1)!} \]

If we compare coefficients of \((n+k)_{2k+1}\) then clearly \(d_{k+1}^{(k+1)} = 0\)

For the remaining \(c_j^{(k+1)}\) and \(d_j^{(k+1)}\) we have, comparing (d) and (f), the recurrences
\[(k+1)^3 c_j^{(k+1)} = B_j^{(k)} c_j^{(k+1)} + C_j^{(k)} c_j^{(k)}\]
\[(k+1)^3 d_j^{(k+1)} = D_j^{(k)} c_j^{(k+1)} + E_j^{(k)} c_j^{(k)}\]

Thus
\[(k+1)^3 c_j^{(k+1)} = \frac{2(j+k)}{(j-1)!^4} (k+j-1)_{2j-2} + \frac{2(j^2 + (k+1)^2)(j+k+1) - (k+1)^3}{(j)!^4} (n+j)_{2j}\]

Noting that
\[(k+j-1)_{2j-2} = \frac{(k+1+j)_{2j}}{(k+j+1)(k+j)} \quad \text{and} \quad (k+j)_{2j} = \frac{k-j+1}{k+j+1}(k+j+1)_{2j}\]

and simplifying the RHS we find that
\[(k+1)^3 c_j^{(k+1)} = \frac{(k+1)^3 (k+1+j)_{2j}}{(j)!^4}\]

From this we easily obtain
\[(k+1)^3 d_j^{(k+1)} = -\frac{2(k+j)(k-j+1)}{(j-1)!^4} (k+j-1)_{2j-2} + \frac{2j^4}{(j)!^4} (k+j)_{2j} = 0\]

Thus we have shown that for \(k \leq n\)
\[q_{n,k} = \binom{n}{k}^3 \tau_{n,k}^{(2)} = \binom{n}{k}^3 (k!)^3 \mu_{n,k} = \binom{n}{k}^3 (k!)^3 \sum_{j=0}^{k} \frac{(n+j)_{2j}(k+j)_{2j}}{(j)!^4} \]

\[\text{................................. (g)}\]

This proves the proposition and as a consequence that \(\mu_{n,k}\) is an integer since each of \((n+j)_{2j}\) and \((k+j)_{2j}\) is a product of \(2j\) consecutive integers and hence divisible by \((j!)^2\). We note that the expression on the RHS is invariant with respect to interchange of \(n\) and \(k\) since when \(j > k\), \((k+j)_{2j} = 0\) hence
\[
\sum_{j=0}^{k} = \min(n,k)
\]

We have already shown elsewhere that \( q_{nk} = q_{nk} \) hence (g) is valid for \( k > n \).

Now we deal with \( p_{nk} \). We have seen that
\[
p_{nk} = (n!)^3 \tau_{n,k}^{(1)} + q_{nk} \Sigma_n
\]
where \( \tau_{n,0}^{(1)} = 0 \) and if \( k \geq 1 \) \( \tau_{n,0}^{(1)} \) is an integral polynomial of degree 2k-2 in \( n \) with \( \tau_{n,1}^{(1)} = 1 \).

Using (5) we obtain
\[
\tau_{n,k+1}^{(1)} = (n+1)^3 \tau_{n+1,k}^{(1)} + \sigma_{n+2,k} \tau_{n,k}^{(1)} + \tau_{n+1,k}^{(2)}
\]
If we write
\[
\tau_{n,k}^{(1)} = (k!)^3 v_{n,k}
\]
we obtain
\[
(k+1)^3 v_{n,k+1} = (n+1)^3 v_{n+1,k} + \sigma_{n+2,k} v_{n,k} + \mu_{n+1,k} \quad \text{..................................} \quad (h)
\]
We note that
\[
v_{n,0} = 0; \quad v_{n,1} = 1; \quad v_{n,2} = \frac{1}{2} \left( \frac{2+1}{(1!)^3} \right) (n+1) + \Sigma_2
\]
\[
v_{n,3} = \frac{1}{3} \left( \frac{3+2}{(2!)^4} \right) (n+2) + \left( \frac{1}{2^3} + \frac{1}{3} \right) \left( \frac{3+1}{(1!)^4} \right) (n+1) + \Sigma_3
\]
suggesting that in general when \( 1 \leq k \leq n \)
\[
v_{n,k} = \sum_{j=0}^{k-1} c_{j}^{(k)} (n+j)_{j} \quad \text{with} \quad c_{j}^{(k)} = \frac{(k+j)_{j}}{(j!)^4} \left[ \Sigma_k - \Sigma_j \right] \quad \text{..................................} \quad (i)
\]
where we define \( \Sigma_0 = 0 \).

We note that the proposition is true when \( k = 1, 2, 3 \). We assume it is true for all values of the variable \( k \) up to some fixed value \( k \). (h) can be transformed to:
\[
(k+1)^3 v_{n,k+1} = (n+1)^3 v_{n+1,k} + \left( 2(n+1)^2(k+1) - 2(n+1)(k+1)^2 + (k+1)^3 \right) v_{n,k} + \mu_{n+1,k} \quad \text{..................................} \quad (j)
\]
and we can write (c.f. (d))
\[
v_{n,k+1} = \sum_{j=1}^{k} c_{j}^{(k+1)} (n+j)_{j} + \sum_{j=1}^{k} d_{j}^{(k+1)} (n+j)_{j-1} + c_{0}^{(k+1)} \quad \text{..................................} \quad (k)
\]
If we substitute (i) into the RHS of (j) we obtain an equation analogous to (f)
\[
(k+1)^3 v_{n,k+1} = \left[ 4k c_{k-1}^{(k)} + \frac{(2k)!}{(k!)^4} \right] (n+k)_{2k} + \left[ -4k c_{k-1}^{(k)} + \frac{2k (2k)!}{(k!)^4} \right] (n+k)_{2k-1} + \sum_{j=1}^{k-1} \left[ B_{j}^{(k)} c_{j}^{(k)} + c_{j}^{(k)} c_{j+1}^{(k)} + \frac{(k+j)_{j} (j!)^4}{(j!)^4} \right] (n+j)_{j} + \sum_{j=1}^{k-1} \left[ D_{j}^{(k)} c_{j}^{(k)} + E_{j}^{(k)} c_{j}^{(k)} + \frac{2j (k+j)_{j} (j!)^4}{(j!)^4} \right] (n+j)_{j-1} + c_{0}^{(k+1)} (k+1)^3 + 1 \quad \text{..................................} \quad (l)
\]
Where A, B, C, D are defined as previously. We can see at once that
\[
c_{0}^{(k+1)} = c_{0}^{(k)} + \frac{1}{(k+1)} \quad \text{hence} \quad c_{0}^{(k)} = \Sigma_k
\]
If we compare coefficients of \((n+k)_{2k}\) in (k) and (l) we have
\[(k+1)^3 c_k^{(k+1)} = 4k c_{k-1}^{(k)} + \frac{(2k)!}{(k!)^3} = \frac{(2k+1)_{2k}}{(k!)^3}\]

and if we compare coefficients of \((n+k)_{2k}\) we readily find that \(d_k^{(k+1)} = 0\)

For the remaining \(c_j^{(k)}\) we have

\[(k+1)^3 c_j^{(k)} = B_j^{(k)} c_{j-1}^{(k)} + C_j^{(k)} c_j^{(k)} + \frac{(k+j)_{2j}}{(j!)^4}\]

and in a manner similar to that used previously we find

\[c_j^{(k+1)} = \frac{(k+1+j)_{2j}}{(j!)^4} \left[ \Sigma_{k+1} - \Sigma_j \right]\]

For the remaining \(d_j^{(k)}\) we have

\[(k+1)^3 d_j^{(k+1)} = D_j^{(k)} c_{j-1}^{(k)} + E_j^{(k)} c_j^{(k)} + \frac{2j(k+j)_{2j}}{(j!)^4}\]

and we easily find that \(d_j^{(k+1)} = 0\)

**Appendix - Shifted factorials**

A shifted factorial \((n)\) is a product of \(j\) consecutive integers, the largest being \(n\), and as such can be seen as a generalisation of the factorial function \(n!\). If \(n < j\) then \((n)\) is zero, otherwise it is \(\frac{n!}{n-j!}\) whence we have \((n)_1 = n, (n)_n = n!\) and can define \((n)_0 = 1\).

Since \(\frac{n!}{n-j!} = j! \binom{n}{j}\) where the binomial coefficient \(\binom{n}{j}\) is an integer, \((n)_j\) is divisible by \(j!\).

We note that when \(j > 0:\)

\[(n+1)_j - (n)_j = j(n)_{j-1}\]

also

\[(n+1)^2 [(n+1+j)_{2j} - (n+j)_{2j}] = 2j(n+1)^3(n+j)_{2j-1}\]

\[= 2j [(n+j+1)_{2j+2} + (j+1)(n+j+1)_{2j+1} + j^2(n+j)_{2j} + j^3(n+j)_{2j-1}] \quad (j > 0)\]

\[2(k+1)(n+1)^2(n+j)_{2j} = 2(k+1) [(n+j+1)_{2j+2} + (n+j+1)_{2j+1} + j^2(n+j)_{2j}]\]

\[= 2(k+1)^2(n+1)(n+j)_{2j} = -2(k+1)^2 [(n+j+1)_{2j+1} - j(n+j)_{2j}]\]

**Lemma C.4:** the continued fraction defined in (11) has the limit \(\zeta(3)\)

**Proof:**

\[
P_n = \sum_{n=0}^{n-1} \left[ \frac{(n+j)_{2j}}{(j!)^2} \right]^2 \left[ \Sigma_{n+1} - \Sigma_j \right] = A_0 + A_1 + \ldots + A_{n-1}
\]

\[
q_n = \mu_{n,n}^{(2)} \sum_{j=0}^{n} \left[ \frac{(n+j)_{2j}}{(j!)^2} \right]^2 = B_1 + B_2 + \ldots + B_n
\]

We show that \(A_j < \frac{B_j}{n}\) and this proves the result.
Lemma C.5 Let \( q_n \) be defined by the recurrence (11). Then \( q_n / q_{n+1} \) is an increasing function of \( n \) and has limit \( 17 + 12\sqrt{2} \)

Proof: From the recurrence relation (11) we obtain

\[
\frac{q_n}{q_{n-1}} \left( \frac{q_{n-1}}{q_{n-2}} \right) - \left( \frac{34 - 51}{n^2} + \frac{27}{n^3} - \frac{5}{n^3} \right) \left( \frac{q_{n-1}}{q_{n-2}} \right) + \left( \frac{n-1}{n} \right)^3 = 0 \quad .................. \quad (a)
\]

So if \( q_n / q_{n-1} \) has a limit it must be a root of \( x^2 - 34x + 1 = 0 \), that is, \( 17 \pm \sqrt{2} \). From (10), we see that \( q_n > q_{n-1} > 0 \) and so if \( q_n / q_{n-1} \) has a limit it must be the larger root since the smaller is less than one. From (a)

\[
\frac{q_n}{q_{n-1}} = \frac{-\left( \frac{n-1}{n} \right)^3}{\left( \frac{q_{n-1}}{q_{n-2}} \right)} + \left( \frac{34 - 51}{n^2} + \frac{27}{n^3} - \frac{5}{n^3} \right) \left( \frac{q_{n-1}}{q_{n-2}} \right)
\]

So \( q_n / q_{n-1} < 34 \)

\[
\frac{q_n}{q_{n-1}} - \frac{q_{n-1}}{q_{n-2}} = \left[ \left( \frac{n-1}{n} \right)^3 - \left( \frac{n-2}{n-1} \right)^3 \right] - 51 \left( \frac{1}{n} - \frac{1}{n-1} \right) + 27 \left( \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) - 5 \left( \frac{1}{n^3} - \frac{1}{(n+1)^3} \right)
\]

If \( q_n/q_{n-1} > q_{n-1}/q_{n-2} \) then both expressions in square brackets are negative and \( q_n/q_{n-1} > q_{n-1}/q_{n-2} \). Since \( q_3/q_2 = 1445/73 > q_2/q_1 = 73/6 \) we have shown inductively that \( q_n/q_{n-1} \) is an increasing function of \( n \) and since it is bounded above it must have a limit.

REFERENCES
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