ANOTHER RECURRENCE FORMULA FOR THE BERNOULLI NUMBERS

Beginning with the generating function for the Bernoulli Numbers we have

\[
\frac{t}{e^t - 1} - \frac{1}{2} = G(t) = \sum_{k=1}^{\infty} \frac{B_{2k}}{2k!} t^{2k}
\]

If we differentiate the left hand side we obtain, after re-arranging,

\[
\left[ G(t) \right]^2 + tG'(t) + G(t) - \left( \frac{1}{2} \right)^2 = 0
\]

Taking the Cauchy product we obtain

\[
\left[ G(t) \right]^2 = \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} \frac{B_{2j} B_{2k-2j}}{2j! 2k-2j!} t^{2k}
\]

also

\[
tG'(t) = \sum_{k=1}^{\infty} \frac{B_{2k}}{2k-1!} t^{2k}
\]

If we allow that coefficients of \( t^{2k} \) sum to zero we have

\[
B_2 = 1/6 \quad (k = 1)
\]

\[
B_{2k} = \frac{-1}{2k+1} \sum_{j=1}^{k-1} \binom{2k}{2j} B_{2j} B_{2k-2j} \quad (k \geq 2)
\]

This result enables us to show inductively that all \( B_{4m} \) are negative and all \( B_{4m+2} \) are positive.

We know that this is true for \( m = 1 \) and \( m = 2 \) respectively (see Sums of Powers ... ). Suppose it is true for all even indexes up to \( 2k - 2 \).

Say \( k \) is even. Then when \( j \) is odd both \( B_{2j} \) and \( B_{2k-2j} \) are positive and when \( j \) is even both \( B_{2j} \) and \( B_{2k-2j} \) are negative.

Now say \( k \) is odd. Then when \( j \) is odd \( B_{2j} \) is positive but \( B_{2k-2j} \) is negative. When \( j \) is even \( B_{2j} \) is negative but \( B_{2k-2j} \) is positive.