SOME PROPERTIES OF THE BERNOULLI POLYNOMIALS

We define the Bernoulli Polynomials (Note 1) with reference to the sum function \( s_{n-1}(x) \) (for the definition and properties of the latter see 'Sums of Powers ....')

\[ \phi_1(t) = t \quad \text{and} \quad \phi_n(t) = n s_{n-1}(t-1) \quad (n \geq 2) \]

Thus

\[ \phi_2(t) = t^2 - t \quad \phi_3(t) = t^3 - \frac{3}{2} t^2 + \frac{1}{2} t \quad \phi_4(t) = t^4 - 2t^3 + t^2 \quad \phi_5(t) = t^5 - \frac{5}{2} t^4 + \frac{5}{3} t^3 - \frac{1}{6} t \]

and in general for \( n > 2 \)

\[ \phi_n(x) = nx - \frac{n}{2} x^{n-1} + \sum_{i=2}^{n-1} \binom{n}{i} B_i x^{n-i} \]

The properties that we are interested in are illustrated below:

Primarily we wish to show that:

(a) \( \phi_{2k}(t) \leq 0 \) for \( 0 \leq t \leq 1 \) and \( k \) odd \( (k > 1) \)
(b) \( \phi_{2k}(t) \geq 0 \) for \( 0 \leq t \leq 1 \) and \( k \) even
(c) \( \phi_{2k+1}(t) \geq 0 \) for \( 0 \leq t \leq 1/2 \) and \( (k \geq 1) \) and \( \phi_{2k+1}(t) \leq 0 \) for \( 1/2 \leq t \leq 1 \) \( (k \) odd \( \geq 1) \)
(d) \( \phi_{2k+1}(t) \leq 0 \) for \( 0 \leq t \leq 1/2 \) and \( (k \geq 1) \) and \( \phi_{2k+1}(t) \geq 0 \) for \( 1/2 \leq t \leq 1 \) \( (k \) even \( \geq 1) \)

but in so doing we show several other properties as well.

First we reprise a number of properties of \( \phi \).

Because \( s_{2k}(t) \) has zeros at \( 0, -1/2, -1 \) \( (k \geq 2) \) we see that \( \phi_{2k+1}(t) \) has zeros at \( 0, 1/2 \) and \( 1 \).

Because \( s_{2k+1}(t) \) has zeros at \( 0, -1 \) \( (k \geq 1) \) we see that \( \phi_{2k}(t) \) has zeros at \( 0 \) and \( 1 \).

We showed in 'Sums of Powers ....' that \(-s_{2k}(y) = s_{2k}(y - 1)\). Putting \( y = 1/2 + x \) we have \( s_{2k}(-1/2 + x) = - s_{2k}(-1/2 - x) \) and consequently \( \phi_{2k+1}(1/2 + t) = - \phi_{2k+1}(-1/2 - t) \) \( (k \geq 1) \).

Using a similar technique we can show that \( s_{2k+1}(-y) = s_{2k}(y - 1) \). Putting \( y = 1/2 + x \) we have \( s_{2k}(-1/2 + x) = s_{2k}(-1/2 - x) \) and consequently \( \phi_{2k}(1/2 + t) = \phi_{2k}(-1/2 - t) \) \( (k \geq 1) \).

Finally, since \( s_k(t-1) - k s_{k-1}(t-1) = B_k \) \( (k \geq 2) \) we have

\[ \phi_{k+1}(t) = (k+1)(\phi_k(t) + B_k) \]

First we show that when \( 0 \leq t \leq 1 \) \( \phi_{2k+1}(t) \) has no zeros other than those at \( 0, 1/2 \) and \( 1 \) and \( \phi_{2k}(t) \) has no zeros other than those at \( 0 \) and \( 1 \) \( (k \geq 1) \).
This is clearly true for $k = 1$ and $2$ since

\begin{align*}
\phi_2(t) &= t(t-1) \\
\phi_3(t) &= t(t-1)(t-\frac{1}{2}) \\
\phi_4(t) &= t^2(t-1)^2 \\
\phi_5(t) &= t(t-1)(t-\frac{1}{2})(t^2-t-\frac{1}{3})
\end{align*}

Now suppose the proposition is true for all indexes up to some even index $k$.

Suppose then that $\phi_{k+1}(t)$ has a zero other than $1/2$ when $0 < t < 1$. If this zero is at $1/2 - t_0$ then there must be a zero at $1/2 + t_0$. Thus there are five distinct zeros in the interval $0 \leq t \leq 1$ and $\phi'(k+1)$ has three distinct zeros in the interval $0 < t < 1$ by Rolle's Theorem (Note 2). So from (2) $\phi_k(t) + B_k$ has three zeros in this interval and so by Rolle's Theorem $\phi'(k)$ has two zeros and thus $\phi_k(t)$ has a zero in the interval $0 < t < 1$, contrary to hypothesis.

Now suppose that $\phi_{k+2}(t)$ has a zero when $0 < t < 1$. If this zero is at $1/2$ then by Rolle's Theorem $\phi'(k+2)$ has zeros in the intervals $0 < t < 1/2$ and $1/2 < t < 1$. But since $B_{k+1} = 0$, by (2) $\phi'(k+1)$ would have zeros at these points also, which we have seen is not the case. If there were a zero at $1/2 - t_0$ then there must be a zero at $1/2 + t_0$. Thus there are five distinct zeros in the interval $0 \leq t \leq 1$ and $\phi'(k+1)$ has three zeros by Rolle's Theorem in the interval $0 < t < 1$ and so from (2) $\phi_{k+1}(t)$ has three zeros also, which we have seen is not the case.

To complete the proofs of (a) - (d) we just need to establish the signs of the $\phi_k(t)$ in the interval $0 < t < 1$.

First consider $\phi_{2k}(t)$. Clearly $\phi_2(t)$ is negative in this interval. Otherwise from (1) we see that $\phi_{2k}(0) = 4k(2k-1)B_{2k-2}$ and thus is negative when $k$ is odd and positive when $k$ is even. If we apply Hardy's 'Theorem A' (see Note 2) twice then we see that for some positive $t_0$ in the vicinity of $t = 0$ then $\phi_{2k}(t)$ is negative when $k$ is odd and positive when $k$ is even for all $0 < t < t_0$. But since since $\phi_{2k}(t)$ has no zeros between $0$ and $1$, this must apply to all $t$ in this interval.

Now consider $\phi_{2k+1}(t)$. From (1) we see that $\phi_{2k+1}(t)$ has the sign of $B_{2k}$ and thus is positive when $k$ is odd and negative when $k$ is even. If we apply Hardy's 'Theorem A' then we see that these signs must also apply to $\phi_{2k+1}(t)$ for all $0 < t < 1/2$. But since since $\phi_{2k}(t)$ has no zeros between $0$ and $1$, this must apply to all $t$ in this interval. Since $\phi_{2k+1}(1/2 + t) = - \phi_{2k+1}(-1/2 - t)$ the opposite must apply for $1/2 < t < 1$.

Note 1 Whittaker and Watson (E T Whittaker and G N Watson A Course of Modern Analysis 4th Edition (1927) reprinted 1965 Chapter VII) define the Bernoulli Polynomials using a generating function. The polynomials thus defined are identical with those defined here. An alternative (and perhaps more 'modern') nomenclature for the polynomials is $B_n(x)$ and the relationship to our $\phi_n(x)$ is $B_1(x) = x - 1/2$ and $B_n(x) = \phi_n(x) + B_n$ ($n > 1$) with $B_n$ being the Bernoulli numbers.

Note 2 'A Course of Pure Mathematics' by G H Hardy, 8th Edition, Cambridge University Press, (1941) Section 122