DESCARTES' CIRCLE THEOREM

If there exist three circles \((C_1, C_2, C_3,\) in black, below) that are mutually tangent externally and have radii \(r_1, r_2, r_3,\) and a fourth circle \((C_4\) in red, below - there are two possibilities) having radius \(r_4\) that is tangent to the first three, then the radii are related by

\[
\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm \frac{1}{r_4}\right)^2 = 2 \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2}\right)
\]

The minus sign is taken if the fourth circle is external to the first three and the plus sign if it is internal.

This is known as Descartes' Circle Theorem (see Note). Here I give two proofs.

The first I found here on the Mathematics Stack Exchange website. It uses vector methods and actually applies to Euclidean spaces of arbitrary dimension. Moreover the author extends his proof to provide a method of constructing the inner and outer spheres.

Let \(d\) be the dimension of the space and \(n = d + 2\) the number of hyperspheres (i.e. four circles when \(d = 2\), five spheres when \(d = 3\) and so on).

Let a vector \(c_i\) give the location of the centre of sphere \(i\) and \(r_i\) be its radius. If \(R = -r\) for the outermost hypersphere and \(R = r\) for the others, the hyperspheres are mutually tangent in the configuration assumed only if

\[
|c_i - c_j| = |R_i + R_j| \quad (i \neq j)
\]

and this is the case only if

\[
c_i \cdot c_i - 2c_i \cdot c_j + c_j \cdot c_j = (R_i + R_j)^2 - 4R_i R_j \delta_{i,j} \quad \text{where } \delta_{i,j} \text{ is the Kronecker delta) ... (1)
\]

Now at least \(n - 2\) of the vectors \(c_2 - c_1, c_3 - c_1 \ldots, c_n - c_1\) span the space, and thus form a basis for the space. When \(d = 2\) at least two of them are not colinear and any vector in the space can be expressed as a linear combination of two of them. When \(d = 3\) at least three of them are not coplanar. Thus the \(n\) th vector may be expressed as a linear combination of the others and we have

\[
\sum_{k=2}^{n} \beta_k (c_k - c_1) = 0 \quad \text{(necessarily at least one } \beta \text{ is non-zero)}
\]

Since \(\beta_1 (c_1 - c_1) = 0\) then \(\beta_1\) can take any value so we put

\[
\beta_1 = -(\beta_2 + \ldots + \beta_n)
\]

and

\[
\sum_{k=1}^{n} \beta_k = 0 \quad \text{and} \quad \sum_{k=1}^{n} \beta_k c_k = 0 \quad \text{.......................................... (2)}
\]
If we multiply (1) by $\beta_i$, then, keeping $j$ fixed, sum over $i$ we get

$$\sum_{i=1}^{n} \beta_i |c_i|^2 - 2c_j \sum_{i=1}^{n} \beta_i c_i + |c_j|^2 \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \beta_i R_i^2 + 2R_j \sum_{i=1}^{n} \beta_i R_i - 4R_j^2 \beta_j$$

Since by (2) the second and third terms on the LHS vanish, this simplifies to

$$4R_j^2 \beta_j = 2AR_j + B \quad \text{where} \quad A = \sum_{i=1}^{n} \beta_i R_i \quad \text{and} \quad B = \sum_{i=1}^{n} \beta_i \left[ R_i^2 - |c_i|^2 \right] \quad ........... (3)$$

If we divide (3) by $R_j$ and sum over $j$ we obtain

$$4A = 2AN + B \sum_{j=1}^{n} \frac{1}{R_j} \quad \text{thus} \quad A = -\frac{B}{2d} \sum_{j=1}^{n} \frac{1}{R_j} \quad ......................................... (4)$$

If we divide (3) by $R_j^2$ and sum over $j$ we obtain

$$4\sum_{j=1}^{n} \beta_j = 0 = 2A \sum_{j=1}^{n} \frac{1}{R_j} + B \sum_{j=1}^{n} \frac{1}{R_j^2} \quad .......................................... (5)$$

Eliminating $A$ from (4) and (5)

$$B \left[ \sum_{j=1}^{n} \frac{1}{R_j^2} - \frac{1}{d} \left( \sum_{j=1}^{n} \frac{1}{R_j} \right)^2 \right] = 0$$

Now $B$ is not zero. By (5) this would imply that $A$ is zero and by (3) $\beta_j$ would be zero for all $j$ which is not the case. Thus

$$d \sum_{j=1}^{n} \frac{1}{R_j^2} = \left( \sum_{j=1}^{n} \frac{1}{R_j} \right)^2$$

which is the desired result.

The second proof is based on one given by Coxeter which is, in turn, a simplification of one given by Beecroft (see Note). Since Coxeter's paper is quite compressed I have made use of proofs given by Jean Pierre Mutanguha here and by L'Hereux et al here, both of which explain the arguments in more detail. Beecroft/ Coxeter's proof, although less elegant than that above, is of interest because it resembles more the 'classical geometry' of Archimedes, Euclid, Apollonius etc.

We first prove two lemmas.

**Lemma 1:** Three circles with centres $O_i$, $O_j$ and $O_k$ and radii $r_i$, $r_j$ and $r_k$ are mutually tangential as shown below. The three points at which they touch define a circle with centre $\Omega_i$ and radius $\rho_i$. We claim that

$$\kappa_i = k_i + k_j k_k + k_i k_j$$

where $\kappa_i = 1/\rho_i$ and $k_i = 1/r_i$ etc

**Proof:** Heron's formula gives the area of a triangle with sides $a$, $b$, $c$ by

$$A^2 = s(s-a)(s-b)(s-c) \quad \text{where} \quad s = (a+b+c)/2$$

but also we see that $A$ is the sum of the areas of triangles $O_i\Omega_i\Omega_j$, $O_i\Omega_i\Omega_k$ and $O_i\Omega_i\Omega_k$.

$$A = \rho_i^2/2 + \rho_j^2/2 + \rho_k^2/2 = s \rho_i^2$$

Putting $a = r_i + r_j$ etc and writing $k_i = 1/r_i$ etc and $\kappa_i = 1/\rho_i$ we obtain the result.
Lemma 2: Three circles with centres $O_i$, $O_j$ and $O_k$ and radii $r_i$, $r_j$ and $r_k$ are mutually tangential as shown below, with circle $k$ circumscribing the first two. The three points at which they touch define a circle with centre $\Omega_h$ and radius $\rho_h$. We claim that
\[ \kappa_h^2 = k_i k_j k_k k_j k_k \] where $\kappa_h = 1/\rho_h$ and $k_i = 1/r_i$ etc

Proof: The method is the same as for Lemma 1. We use Heron's formula to obtain the area $A$ of triangle $O_j O_k O_i$, noting that
\[ O_j O_k = r_k - r_j \quad O_k O_i = r_k - r_i \quad O_j O_i = r_j + r_i \]
This gives
\[ A^2 = r_i r_j r_k (r_k - r_j - r_i) \]
but also we see that $A$ is the area of the quadrilateral $\Omega_h T_1 O_k T_3$ less the area of the pentagon $\Omega_h T_1 O_j O_i T_3$ and the latter is the sum of the areas of quadrilaterals $\Omega_h T_1 O_j T_2$ and $\Omega_h T_2 O_i T_3$ so
\[ A = \rho_h r_k - \rho_h r_j - \rho_h r_i = \rho_h (r_k - r_j - r_i) \]
which gives the desired result.
Returning to the Descartes Circles we consider first the case where the $C_4$ is interior to $C_{1-3}$. For every three of the four circles $C_{1-4}$ (in black, below) there exists a circle passing through the points at which they touch (blue circles, below). Using Lemma 1 and assigning $\kappa_h$ to the blue circle which passes through the points at which black circles $C_i, C_j$ and $C_k, i, j, k \neq h$ touch we see that

$$\kappa_h^2 = k_1 k_j + k_j k_k + k_k k_i$$

where $h, i, j, k = 1, 2, 3, 4$ and $h \neq i \neq j \neq k$ ......... (1)

thus

$$\sum_{n=1}^{4} \kappa_n^2 = 2 \left( k_1 k_2 + k_1 k_3 + k_1 k_4 + k_2 k_3 + k_2 k_4 + k_3 k_4 \right)$$

but also

$$\left( \sum_{n=1}^{4} k_n \right)^2 = \sum_{n=1}^{4} \kappa_n^2 + 2 \left( k_1 k_2 + k_1 k_3 + k_1 k_4 + k_2 k_3 + k_2 k_4 + k_3 k_4 \right)$$

Thus

$$\left( \sum_{n=1}^{4} k_n \right)^2 = \sum_{n=1}^{4} \kappa_n^2 + \sum_{n=1}^{4} \kappa_n^2$$

But each of the black circles passes through the points at which three of the blue circles touch. Thus applying the Lemmas we have (noting that we have allocate the index 4 to the circumscribing blue circle)

$$k_4^2 = k_1 k_2 + k_2 k_3 + k_3 k_1 \quad \text{and} \quad k_h^2 = k_1 k_2 - k_4 k_1 - k_4 k_2 \quad \text{where} \quad h = 1, 2, 3 \quad \text{and} \quad h \neq i \neq j \quad ... \quad (2)$$

Thus

$$\sum_{n=1}^{4} k_n^2 = 2 \left( k_1 k_2 + k_1 k_3 + k_2 k_3 \right) - 2 \left( k_1 k_4 + k_2 k_4 + k_3 k_4 \right) \quad \text{..................................} \quad (2a)$$

but also

$$\left( \kappa_1 + \kappa_2 + \kappa_3 - \kappa_4 \right)^2 = \sum_{n=1}^{4} \kappa_n^2 + 2 \left( \kappa_1 k_2 + \kappa_1 k_3 + \kappa_2 k_3 \right) - 2 \left( \kappa_1 k_4 + \kappa_2 k_4 + \kappa_3 k_4 \right)$$

So

$$\left( \kappa_1 + \kappa_2 + \kappa_3 - \kappa_4 \right)^2 = \sum_{n=1}^{4} \kappa_n^2 + \sum_{n=1}^{4} \kappa_n^2 = \left[ \sum_{n=1}^{4} k_n \right]^2 \quad \text{..................................} \quad (3)$$
To complete the proof for this case, we need to show that
\[ \sum_{n=1}^{4} k_n^2 = \sum_{n=1}^{4} \kappa_n^2 \]
We note that as the circumscribing blue circle has the index 4, then \( \kappa_4 < \kappa_1, \kappa_2, \kappa_3 \) thus from (3)
\[ \kappa_1 + \kappa_2 + \kappa_3 - \kappa_4 = \sum_{n=1}^{4} k_n \] ................................................................. (4)
Now
\[-(k_1 + k_2 + k_3 + k_4)\left(-k_1 + k_2 + k_3 + k_4\right) = -k_1^2 + k_2^2 + k_3^2 + k_4^2 + 2\left(k_2 k_3 + k_2 k_4 + k_3 k_4\right) \]
by (1) \[ = -k_1^2 + k_2^2 + k_3^2 + k_4^2 + 2 \kappa_1^2 = -2 k_1^2 + \left[ \sum_{n=1}^{4} k_n^2 \right] + 2 \kappa_1^2 \]
by (2) and (2a) \[ = 2 \kappa_1 \left( \kappa_1 + \kappa_2 + \kappa_3 - \kappa_4 \right) = 2 \kappa_1 \left( k_1 + k_2 + k_3 + k_4 \right) \]
So using (4) \[-k_1 + k_2 + k_3 + k_4 = 2 \kappa_1 \]
In similar fashion
\[ k_1 - k_2 + k_3 + k_4 = 2 \kappa_2 \]
\[ k_1 + k_2 - k_3 + k_4 = 2 \kappa_3 \]
\[ k_1 + k_2 + k_3 - k_4 = -2 \kappa_4 \]
Squaring each of these four equations and adding we find
\[ \sum_{n=1}^{4} k_n^2 = \sum_{n=1}^{4} \kappa_n^2 \]
which completes the proof of Descartes' Circle Theorem for this case. The case where \( C_4 \) is the exterior circle is also covered by the proof. We take \( C_{1,4} \) to be the blue circles. If we exchange the \( k_n \) and the \( \kappa_n \) we see by (3) that
\[ [\kappa_1 + \kappa_2 + \kappa_3 - \kappa_4]^2 = 2 \sum_{n=1}^{4} \kappa_n^2 \]
Thus we have incidentally proved this case also.

Note
Lagarias et al (Amer. Math. Monthly Volume 109 (2002), pages 338-361, available in pdf form here) give some historical background. The theorem was first stated, without full proof, by Descartes in 1643. Proofs were published independently in 1826 by Jakob Steiner and in 1842 by Philip Beecroft. The papers of Steiner (in German) cited by Lagarias et al may be found via this link. Coxeter's 1968 paper, which is referenced by Lagarias et al and in the links above, is not directly accessible in electronic form but a copy may be obtained via Jstor (see here). Beecroft's paper was published in the Lady's and Gentleman's Diary, and may be accessed via this link.