LAMBERT'S IRRATIONALITY PROOFS FOR e AND π

Johann Heinrich Lambert was a Swiss mathematician of the mid 18th century who is credited with the first proof that π is irrational (Note 1).

The original of Lambert's 1761 paper, in French, together with a commentary on it by Alain Juhel, in both French and English translation, may be found here. Another authoritative, and very well referenced, discussion by Dave L Renfro may be found here.

What I write here is based on the presentation of Lambert’s continued fraction method given in sections 20 and 21 of Chrystal’s ‘Algebra’ (Note 2).

We use the criterion (Note 3) that

\[
\text{if there exist integers } h \text{ and } k \text{ such that } 0 < |kf - h| < \varepsilon \text{ for any given } 1 > \varepsilon > 0 \text{ then } f \text{ is irrational}
\]

This suggests we should look for sequences of integers \(h_n, k_n\) such that \(|k_n f - h_n| \text{ diminishes with increasing } n\). Lambert achieved more. He found sequences \(h_n(x)\) and \(k_n(x)\) for \(f = f(x) = x^{-1} \tan h x \text{ and } f(x) = x^{-1} \tan x\) in the form of continued fractions, and used them to show that \(\tan h x\) and \(\tan x\) are irrational for all non-zero rational \(x\). By taking \(x = 1\) and \(x = \pi/4\) respectively he showed that both \(e\) and \(\pi\) are irrational.

The continued fractions are

\[
\frac{\tanh x}{x} = \frac{1}{1 + \frac{x^2}{3 + \frac{x^2}{5 + \frac{x^2}{\ldots}}}} \quad \text{and} \quad \frac{\tan x}{x} = \frac{1}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \frac{x^2}{\ldots}}}}
\]

Continued fractions and irrationality criteria

Continued fraction representations of a function take the following general form

\[
f(x) = s_1(x) = \frac{a_1(x)}{b_1(x) + s_2(x)} \quad \text{and} \quad s_k(x) = \frac{a_k(x)}{b_k(x) + s_{k+1}(x)} \quad (k \geq 2) \quad \text{............... (1)}
\]

where \(a_k(x)\) and \(b_k(x)\) are ratios of polynomials in \(x\) with rational coefficients. That is to say, they have the form

\[
c_0 + c_1 x + \ldots + c_n x^n
\]

\[
d_0 + d_1 x + \ldots + d_n x^n
\]

with \(c_k, d_k\) rational. If we define

\[
p_{-1} = 1 \quad q_{-1} = 0 \quad p_0 = 0 \quad q_0 = 1
\]

and

\[
p_{k+1}(x) = a_{k+1}(x)p_{k-1}(x) + b_{k+1}(x)p_k(x) \quad q_{k+1}(x) = a_{k+1}(x)q_{k-1}(x) + b_{k+1}(x)q_k(x) \quad (k \geq 0) \quad \text{............... (2)}
\]

then (Note 4)

\[
f(x) = \frac{p_n(x) + p_{n-1}(x)s_{n+1}(x)}{q_n(x) + q_{n-1}(x)s_{n+1}(x)} \quad (n \geq 0) \quad \text{............... (3)}
\]

with \(p_n(x)\) and \(q_n(x)\) necessarily also ratios of polynomials with rational coefficients. Using (2) and (3) and abbreviating \(p_n(x)\) as \(p_n\) etc we obtain

\[
q_n f(x) - p_n = \frac{[q_n p_{n-1} - p_n q_{n-1}]s_{n+1}}{q_n + q_{n-1}s_{n+1}} = \frac{(-1)^n - [a_n a_{n-1} \ldots a_1]s_{n+1}}{q_n + q_{n-1}s_{n+1}} \quad \text{............... (4)}
\]

The original of Lambert's 1761 paper, in French, together with a commentary on it by Alain Juhel, in both French and English translation, may be found here. Another authoritative, and very well referenced, discussion by Dave L Renfro may be found here.
Now let $\lambda_0(x) = 1, \lambda_1(x), \lambda_2(x), \ldots$ be a sequence of ratios of polynomials with rational coefficients. Define $a'_j = \lambda_j a_{j-1}, b'_j = \lambda_j b_j, s'_{n+1} = \lambda_n s_n + 1$. Then it is easy to show that the continued fraction formed by replacing the unprimed quantities in (1) with the primed quantities is also equal to $f(x)$. Thus (4) becomes

$$q'_{n}f(x) - p'_{n} = \frac{(-1)^{n+1}a_n a'_{n-1} \ldots a'_{1}s'_{n+1}}{q'_{n} + q'_{n-1}s'_{n+1}} = R'_{n}(x) \quad \text{..........................} \quad (5)$$

and it is easy to see using (2) that

$$p'_{k} = \lambda_{1} \lambda_{2} \ldots \lambda_{k} p_{k} \quad \text{and} \quad q'_{k} = \lambda_{1} \lambda_{2} \ldots \lambda_{k} q_{k} \quad (k \geq 1)$$

We can choose the $\lambda_n(x)$ such that the $a'_n(x)$ and $b'_n(x)$ are integral polynomials in $x$ (that is, polynomials whose coefficients are integers). Consequently, by (2), $q'_n(x)$ and $p'_n(x)$ are integral polynomials. If $d(n)$ is the degree of whichever of $q'_n(x)$ or $p'_n(x)$ has the higher degree, and if $x = u/v$ is rational with $u, v$ being integers having greatest common factor 1, then $v^{d(n)}q'_n(x)$ and $v^{d(n)}p'_n(x)$ are integers and the criterion applies if we can show that for any given rational $x$ we can find an $n$ such that

$$0 < |v^{d(n)}R'_{n}(x)| < \varepsilon$$

for any given $0 < \varepsilon < 1$.

**Constructing Lambert's continued fractions**

$$\frac{\tan x}{x} = \frac{\sinh x}{x} \frac{1}{\cosh x} = \frac{e^x - e^{-x}}{x} \frac{1}{e^x + e^{-x}} = \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{x^{2k}}{(2k)!} = \frac{f_2(x)}{f_1(x)}$$

$$\frac{\tan x}{x} = \frac{\sin x}{x} \frac{1}{\cos x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = \frac{f_2(x)}{f_1(x)}$$

where $f_2(x)$ and $f_1(x)$ are even Maclaurin series of the form

$$f_n(x) = \sum_{k=0}^{\infty} f_n^{(2k)} \frac{x^{2k}}{(2k)!} \quad \text{..............................................................} \quad (6)$$

In the present case the series for $f_1(x)$ and $f_2(x)$ are absolutely convergent and as a consequence (while recognising the possibility of a zero denominator) the following is valid

$$f_2(x) - \frac{2}{f_1^{(0)}} f_1(x) = x^2 f_3(x)$$

with $f_3(x)$ also an absolutely convergent even Maclaurin series. We can repeat this process indefinitely to obtain a sequence of equations

$$f_{n+1}(x) - \frac{f_{n+1}^{(0)}}{f_n^{(0)}} f_n(x) = x^2 f_{n+2}(x) \quad (n \geq 1) \quad \text{..........................} \quad (7)$$

If we put

$$s_n(x) = f_{n+1}(x)/f_n(x)$$

We can re-write this as

$$s_n(x) = \frac{-s_n(0)/x^2}{-1/x^2 + s_{n+1}(x)} \quad (n \geq 1)$$
This is a continued fraction of the form (1) with \( a_k = -s_n(0)/x^2 \) and \( b_k = -1/x^2 \)

If we substitute the series (6) into (7) and equate coefficients of equal powers of \( x \) we obtain

\[
f_n^{(2k)} = \frac{1}{(2k+1)(2k+2)} \left( f_n^{(2k+2)} - f_n^{(2k+1)} s_n(0) \right)
\]

(8)

from which we are able to iteratively calculate \( s_n(0) \)

**tanh x**

In this case

\[
f_1^{(2k)} = 1, \quad f_2^{(2k)} = \frac{1}{2k+1}
\]

Calculating \( f_n^{(2k)} \) for \( n = 1, 2, 3 \) using (8) suggests that

\[
f_n^{(2k)} = \frac{(-1)^n}{(2k+1)(2k+3)\ldots(2k+2n-3)1.3\ldots2n-5}
\]

(\( n \geq 3 \))

and this can be verified by induction on \( n \), using (8). So

\[
s_1(0) = 1, \quad s_n(0) = \frac{f^{(0)}_{n+1}}{f^{(0)}_n} = \frac{-1}{(2n-1)(2n-3)}
\]

and we have the continued fraction

\[
s_1(x) = \frac{\tanh x}{x} = \frac{-1/x^2}{-1/x^2 + s_2(x)}
\]

\[
s_k(x) = \frac{1/[\{(2k-3)(2k-1)x^2\}]}{-1/x^2 + s_{k+1}(x)}
\]

(\( k \geq 2 \))

If we apply the transformation described above with \( \lambda = -(2k - 1)x^2 \) (\( k \geq 1 \)) we obtain Lambert's continued fraction for \( \tanh x/x \)

\[
s'_1(x) = \frac{\tanh x}{x} = \frac{1}{1 + s'_2(x)}
\]

\[
s'_k(x) = \frac{x^2}{(2k-1) + s'_{k+1}(x)}
\]

(\( k \geq 2 \))

Since \( a_k' = x^2 \) and \( b_k' = 2k - 1 \) are integral polynomials in \( x \), this yields an expression of the form (5), namely

\[
q'_{n}(x) \left[ x^{-1} \tanh x \right] - p'_{n}(x) = \frac{(-1)^{n-1}x^{2n-2}s'_{n+1}(x)}{q'_{n}(x) + q'_{n-1}s'_{n+1}(x)} = R'_{n}(x)
\]

Using (2) and induction the degree of the integral polynomials \( p'_{n}(x) \) and \( q'_{n}(x) \) are easily seen to be \( \leq n \). Therefore we can apply the criterion by showing that for any given integer \( v \) we can find an \( n \) such that \( 0 < |v^n R'(x)| < \varepsilon \).

First we show that \( s_{n+1}(x) \) is positive but can be made as small as we like. We have

\[
s_{n+1}(x) = \frac{f_{n+2}(x)}{f_{n+1}(x)} = \sum_{k=0}^{\infty} f_{n+2}^{(2k)} \frac{x^{2k}}{2k!(2k)!} - \sum_{k=0}^{\infty} f_{n+1}^{(2k)} \frac{x^{2k}}{2k!(2k)!} = \frac{1}{2n-1} \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)\ldots(2k+2n-3)(2k)!} x^{2k}
\]

\[
\sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)\ldots(2k+2n-3)(2k)!} x^{2k}
\]

(\( n \geq 3 \))
Since
\[
\frac{(2k+1)(2k+3)\ldots(2k+2n-1)}{(2k+1)(2k+3)\ldots(2k+2n+1)} < \frac{1}{2n+1}
\]
we have
\[
0 > s_{n+1}(x) > \frac{-1}{(2n-1)(2n+1)}
\]
and this can be verified by induction on \(n\), using (8). So

If \(x = 0\) then \(R_n'(x) = 0\) and the irrationality criterion cannot apply (we know in any case that \(x^t \tanh x = 1\) when \(x = 0\)). Otherwise let \(x = u/v\) be rational with \(u \neq 0\) and suppose \(u, v\) have no common factors. Then
\[
0 < |v^n R_n(x)| < \frac{u^{2n}}{(2n+1)(2n-1)\ldots3.1} < Kr^n
\]
for sufficiently large \(n\) and where \(K, r\) are independent of \(n\), \(K > 0\) and \(0 < r < 1\). Therefore \(x^t \tanh x\) is irrational if \(x\) is non-zero and rational since \(Kr^n\) can be made arbitrarily small in absolute value by making \(n\) large enough. Clearly if \(x^t \tanh x\) is irrational for non-zero, rational \(x\) then so is \(\tanh x\).

If in particular we take \(x = 1\) and note that
\[
\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \text{hence} \quad \tanh 1 = \frac{e^2 - 1}{e^2 + 1}
\]
we see that \(e^2\) must be irrational, because if it were rational \(\tanh 1\) would be rational, a contradiction. Furthermore \(e\) must be irrational, because if it were rational \(e^2\) would be rational.

\(\tan x\)

The treatment of \(\tan x\) closely follows that of \(\tanh x\). However because series and continued fractions with terms alternating in sign are involved, there is greater difficulty in arriving at an upper bound on the absolute value of the remainder term \(R_n'(x)\).

We have
\[
f_1^{(2k)} = (-1)^k \quad f_2^{(2k)} = \frac{(-1)^k}{2k+1}
\]
Calculating \(f_n^{(2k)}\) for \(n = 1, 2, 3\) using (8) suggests that
\[
f_n^{(2k)} = \frac{(-1)^k}{(2k+1)(2k+3)\ldots(2k+2n-3)\ldots2n-5} \quad (n \geq 3) \quad \text{............ (9)}
\]
and this can be verified by induction on \(n\), using (8). So
\[ s_1(0) = 1 \quad \text{and} \quad s_n(0) = \frac{f_{n+1}^{(0)}}{f_n^{(0)}} = \frac{1}{(2n-1)(2n-3)} \]

and we have the continued fraction
\[ s_1(x) = \tan x \quad \frac{x}{x - 1/x^2 + s_2(x)} \]
\[ s_k(x) = \frac{-1/((2k-3)(2k-1)x^2)}{-1/x^2 + s_{k+1}(x)} \quad (k \geq 2) \]

If we again apply the transformation \( \lambda_k = -(2k-1)x^2 \) (\( k \geq 1 \)) we obtain Lambert's continued fraction for \( x^2 \tan x \)
\[ s_1(x) = \tan x \quad \frac{1}{1 + s_2'(x)} \]
\[ s_k'(x) = \frac{-x^2}{(2k-1) + s_{k+1}'(x)} \quad (k \geq 2) \]

Since \( a_k' = -x^2 \) and \( b_k' = 2k -1 \) are integral polynomials in \( x \), this yields an expression of the form (5), namely
\[ q'_n(x) | x^{-1} \tan x | - p'_n(x) = \frac{x^{2n-2}s'_{n+1}(x)}{q'_n(x) + q'_{n-1}s'_{n+1}(x)} = R'_n(x) \quad \text{...............} (10) \]

The degrees of \( q'_n(x) \) and \( p'_n(x) \) are \( \leq n \). To apply the criterion we need to show that for any given integer \( v \) we can find an \( n \) such that \( 0 < |vR_n(x)| < \varepsilon \).

Again we first examine the magnitude of \( |s_{n+1}(x)| \)
\[ s_{n+1}(x) = \frac{f_{n+2}(x)}{f_{n+1}(x)} = \sum_{k=0}^{n} \frac{f_{n+2}^{(2k)} x^{2k}}{(2k)!} = \frac{1}{2n-1} \sum_{k=0}^{n} \frac{(-1)^k (2k+1)(2k+3)...(2k+2n+1)}{(2k)!} x^{2k} \]

Both numerator and denominator are alternating series of the form \( a_0 - a_1 + a_2 - \ldots \) with \( a_k > 0 \). For the series in the numerator \( a_{n+1}/a_k = x^2/(2k+2n+3)(2k+2) \). We are permitted to take \( n \) indefinitely large and if \( n > x^2 \) the sequence \( a_0, a_1, \ldots \) is monotone decreasing and the sum of an alternating series of this type lies between \( a_0 \) and \( a_0 - a_2 \) (Note 5). So for sufficiently large \( n \) the absolute value of the sum in the numerator is less than \( 1/[1.3\ldots(2n+1)] \). In a similar manner we find that for \( n > x^2 \) the sum in the denominator is greater than
\[ \frac{1}{1.3\ldots(2n-1)} \left[ 1 - \frac{x^2}{2(2n+1)} \right] > \frac{3}{4} \frac{1}{1.3\ldots(2n-1)} \]

So for sufficiently large \( n \)
\[ 0 < \frac{f_{n+2}(x)}{f_{n+1}(x)} < \frac{4}{3} \frac{1}{(2n-1)(2n+1)} \]

and
\[ 0 > s'_{n+1} > -\frac{4}{3} \frac{x^2}{2n+1} \]

We turn now to the denominator of (10). Numerical experimentation suggests that depending on the value of \( x^2 \), for sufficiently large \( k \) either \( q_k > q_{k+1} > 0 \) or \( q_k < q_{k+1} < 0 \) and we verify this
algebraically in Note 6.

Using (2) we have

\[ q_{k+2} - q_{k+1} = (2k+2)q_{k+1} - 2xq_{k} = (2k+2)(q_{k+1} - q_{k}) + (2k + 2 - x^2)q_{k} \]

We can take \( k \) sufficiently large that \( 2k + 2 - x^2 > 0 \). Therefore for \( m \geq 0 \), either (if \( q_{k} > 0 \))

\[ q_{k+2+m} - q_{k+1+m} = (2k+2)(q_{k+1+(m-1)} - q_{k+1+(m-1)}) > (2(k+m)+2)(2(k+m))...(2k+2) > 0 \]

or (if \( q_{k} < 0 \))

\[ q_{k+2+m} - q_{k+1+m} = (2k+2)(q_{k+1+(m-1)} - q_{k+1+(m-1)}) < (2(k+m)+2)(2(k+m))...(2k+2) < 0 \]

but in any case for large enough \( n \)

\[ \left| q_{n} + q_{n-1}s'_{n+1}(x) \right| > \left| q_{n-1}(1 + s'_{n+1}(x)) \right| \]

and

\[ \left| q_{n} \right| > (2n - 2)c_{1} + c_{2} \]

where \( c_{1} > 0 \) and \( c_{2} \) are independent of \( n \). Since \( s'_{n+1} \) can be made arbitrarily small by taking \( n \) large enough, this is sufficient to show that \( \left| v_{n}R_{n}(x) \right| \) can be made arbitrarily small in absolute value, and thus for non-zero rational \( x \), \( \tan x \) is irrational. Consequently since \( \tan \pi/4 = 1 \) is rational, then \( \pi/4 \) and hence \( \pi \) is irrational.

Notes

[1] See, for example, the end-notes of Chapter 4 of Hardy and Wright *An Introduction to the Theory of Numbers* (5th edition)

[2] George Chrystal *Algebra. An elementary text-book* Adam & Charles Black, 1900. This is available as two bulky pdfs at onlinebooks

[3] Since \( |kx - h| \neq 0, h/k \neq x \). Since \( e < 1 \) then \( k \neq 0 \). The rest follows by the contrapositive to the following proposition:

\[ \text{if } x \text{ is rational then there exists a positive integer } n \text{ such that for every pair of integers } h, k \text{ where } k \neq 0 \text{ and } h/k \neq x, |kx - h| \geq 1/n, \text{ with } n \text{ being independent of } h, k \]

The proposition is true if \( x = 0 \) because then \( |kx - h| = |h| \geq 1 \). Otherwise suppose \( x = a/b \) with \( a, b \neq 0 \) and \( (a, b) = 1 \). Then \( |kx - h| = |ka - hb|/|b| \). The numerator is an integer and cannot be zero because if it were, we would have \( ka - hb = 0 \) and \( h/k = a/b = x \). Therefore the numerator is \( \geq 1 \). Identifying \( |b| \) (which is uniquely determined by \( x \)) with \( n \) we have the result.

[4] When \( n = 0 \), the formula gives the identity \( s_{1}(x) = s_{1}(x) \). When \( n = 1 \) we obtain

\[ s_{1}(x) = \frac{p_{1}(x)}{q_{1}(x) + s_{2}(x)} = \frac{a_{1}(x)}{b_{1}(x) + s_{2}(x)} \]

Suppose the proposition is true for \( n \leq m \). Then substituting

\[ s_{m+1}(x) = \frac{a_{m+1}(x)}{b_{m+1}(x) + s_{m+2}(x)} \]

in the expression

\[ s_{1}(x) = \frac{p_{m}(x) + p_{m-1}(x)s_{m+1}(x)}{q_{m}(x) + q_{m-1}(x)s_{m+1}(x)} \]
we easily see the proposition is true for \( n = m + 1 \).


[6] The proposition is true if \( q_k/q_{k-1} > 1 \) for all \( k \) greater than some sufficiently large value.

The sequence \( q_k \) does not contain more than one zero in succession. By (2)

\[
q_{-1}=0 \quad q_0=1 \quad q_1=1 \quad \text{and} \quad q_{k+2}(x)=-x^2q_k(x)+(2k-1)q_{k+1}(x) \quad (k \geq 0)
\] ................................. (a)

Suppose there are no successive zeros for any subscript \( \leq K+1 \). If \( q_K = 0 \) then \( q_{K+1} \neq 0 \) by hypothesis so \( q_{K+1} \) and \( q_{K+2} \) are not successive zeros. If \( q_{K+1} = 0 \) then \( q_K \neq 0 \) and since \( x \neq 0 \) then \( q_{K+2} = -x^2q_K \neq 0 \). One consequence of this result is that for any given index \( K \), there is always a \( q_{K+m} \) \((m > 0)\) that is non-zero.

Now given some \( x \), we find a \( K > 1 \) such that \( 2K - 1 > x^2 \) and \( q_{K+1} \neq 0 \)

Then if \( q_K = 0 \) \( q_{K+2}/q_{K+1} = 2K - 1 > 1 \)

If \( q_K \neq 0 \) then

\[
q_{K+2}/q_{K+1} = -x^2 + (2K - 1) \]

................................. (b)

and if \( q_{K+1}/q_K \geq 1 \) then \( q_{K+2}/q_{K+1} > 1 \) and subsequently all \( q_{K+m+2}/q_{K+m+1} > 1 \)

Now suppose that \( q_{K+1} \) and \( q_K \) have opposite signs and \( q_K \neq 0 \). Then by (b) \( q_{K+2}/q_{K+1} > 1 \)

There remains the case \( 0 < q_{K+1}/q_K < 1 \). From (b)

\[
q_{K+3}/q_{K+2} = 2 - x^2 \left[ \frac{1}{q_{K+2}/q_{K+1}} - \frac{1}{q_{K+1}/q_K} \right]
\]

If \( q_{K+1}/q_K \leq q_{K+2}/q_{K+1} \) then

\( q_{K+4}/q_{K+2} \geq 2 + q_{K+2}/q_{K+1} > 1 \)

and \( q_{K+m}/q_{K+m-1} > 1 \) for all \( m \geq 2 \).

Otherwise \( q_{K+2+m}/q_{K+1+m} < q_{K+1+m}/q_{K+m} \) for all \( m > 0 \). Then necessarily \( q_{K+1+m}/q_{K+m} \) approaches some limit \( L \geq 0 \).

If \( L > 0 \), we could make \( q_{K+2}/q_{K+1} + \frac{x^2}{q_{K+1}/q_K} \) arbitrarily close to the value (independent of \( K \)) \( L+x^2/L \) by taking \( K \) large enough, but since this expression (see (b)) necessarily approaches \( 2K + 1 \), this is impossible.

If \( L = 0 \), then for any given real number \( \varepsilon > 0 \), there exists a \( K(\varepsilon) \) such that for all \( m \geq 0 \), \( q_{K+m+1}/q_{K+m} < \varepsilon \). Taking \( \varepsilon < 1 \) then \( |q_{K+m+1}| < \varepsilon |q_{K+m}| < \varepsilon^2 |q_{K+m-1}| < \ldots < \varepsilon^n |q_K| \). By taking \( m \) sufficiently large we can make \( \varepsilon^n < 1/|q_K| \) and therefore \( |q_{K+m+1}| < 1 \), which is impossible.

Thus in all possible cases, the proposition is true.